African Journal of Advanced Pure and Applied Sciences (AJAPAS)
Online ISSN: 2957-644X
Volume 4, Issue 3, 2025
Page No: 462-471
Website: https://aaasjournals.com/index.php/ajapas/index
ISI 2025: 1.126 SJIFactor 2024: 6.752 معامل التأثير العربي: 1.62

Conformal Mapping Methods for Solving Boundary Value Problems in Complex Domains

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طرق التحويل المطابق لحل مسائل القيم الحدية في المجالات المركبة

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Received: July 07, 2025	Accepted: August 26, 2025	Published: September 08, 2025

Abstract:
This study focuses on the fundamental role that conformal transformations play in complex analysis due to their angle-preserving properties, making them particularly important in both pure mathematics and applied fields such as physics and engineering. This paper investigates the mathematical principles of conformal mappings, starting with a review of harmonic and analytic functions, and their relationship through the Cauchy-Riemann equations. A conformal mapping is formally defined as a transformation w=f(z) that preserves both the magnitude and orientation of angles between curves in the complex plane, provided that f(z) is analytic and its derivative does not vanish. Representative examples of conformal mappings, including linear transformations, rotations, translations, and power mappings are presented. In addition, it demonstrates how these mappings simplify complex geometric problems by preserving essential geometric properties and enabling solutions across different coordinate systems. The paper briefly contrasts conformal mappings with isogonal mappings, preserving angles but not orientations.

Keywords: Conformal transformations, Complex analysis, Analytic function, Cauchy-Riemann equations, Angle-Preserving property.

الملخص
يهدف هذا البحث إلى إبراز الدور الجوهري الذي تؤديه التحويلات المطابقة في التحليل المركب، نظراً لخاصيتها في حفظ الزوايا، مما يجعلها ذات أهمية خاصة في الرياضيات البحتة وتطبيقاتها في الفيزياء والهندسة. يستعرض البحث المبادئ الرياضية للتحويلات المطابقة، مبتدئاً بمراجعة للدوال التوافقية والتحليلية والعلاقة بينها من خلال معادلات كوشي- ريمان. ويعرف التحويل المطابق بأنه تحويل يحافظ على مقدار واتجاه الزوايا بين المنحنيات في المستوى المركب، بشرط أن يكون دالة تحليلية مشتقتها لا تساوي الصفر. كما تُعرض أمثلة رئيسية لهذه التحويلات، مثل التحويلات الخطية، تحويلات الدوران، تحويلات الانتقال، وتحويلات القوى، وذلك لتوضيح هذه المفاهيم. ويُبين كيف تسهل هذه التحويلات مُعالجة المسائل الهندسية المعقدة من خلال الحفاظ على الخصائص الهندسية الأساسية وإتاحة حلول عبر أنظمة إحداثية مختلفة. ويتناول البحث بإيجاز المقارنة بين التحويلات المطابقة والتحويلات متساوية الزوايا، التي تحافظ على الزوايا دون أن تحافظ على اتجاهها.

الكلمات المفتاحية: التحويلات المطابقة، التحليل المركب، الدوال التحليلية، معادلات كوشي- ريمان، خاصية حفظ الزوايا.

Introduction

Conformal mappings are bijective holomorphic functions that preserve angles locally where their derivative is nonzero [6,12, 16,17]. Building on the work of Gauss and Riemann, Cauchy's formulation via the Cauchy–Riemann equations give necessary and sufficient conditions for analyticity and hence conformality in the plane.

Standard expositions [1,7] show how such maps recast boundary-value problems for Laplace's equation by transporting intricate domains to canonical ones—most notably the unit disc—without altering essential features. A key subclass is the Möbius transformations, which act on the Riemann sphere, preserve circles and lines, and decompose into translations, rotations, dilations, and inversions [2,14]. These tools underpin applications in electrostatics, fluid dynamics, and aerodynamics, with growing computational use in solving PDEs where angle preservation supports physical fidelity [6,17]. Several lectures and online resources serve as supplementary materials that facilitate the understanding of conformal mapping concepts [8,13,14].

The Riemann Mapping Theorem is a cornerstone, asserting that any two simply connected, proper subdomains of the complex plane are conformally equivalent, which is pivotal for simplifying problems like Dirichlet problems by transforming domains [12,16]. Key examples include linear, power, and exponential. Overall, conformal mapping remains a vibrant interface between classical complex function theory and modern computational methods. However, integrating classical conformal mapping theory with modern computational methods to handle dynamically changing physical domains is still an open area.

This study intends to contribute to this field by investigating an enhanced conformal mapping framework that combines rigorous mathematical theory with computational techniques, enabling efficient solutions to boundary value problems in complex geometries.

Preserving angles is an important geometric property in complex analysis because when changing variables, it is important not to alter the physical significance of a problem. For example, suppose $w = f(z)$ maps a domain $D($ in $xy - plane)$ with boundary ∂D into another domain $\mathbb{D}$ $(in\, uv - plane)$ with boundary $\partial\mathbb{D}$. This mapping transforms the problem from $z - plane$ into a simpler problem $w - plane$. If the problem can be solved in the new domain, then the transformation comes back as an invertible mapping to give the solution of this problem in the original one. However, in general, invertible mappings do not preserve geometric statements of the coordinate system. This is a crucial point, since geometrically a complex number z is viewed as a vector represented by (x, y) in $z - plane$, therefore to measure the magnitude of this vector we represent

$$|\vec{z}| = \sqrt{x^2 + y^2}$$

, this is true in Euclidean coordinate (x, y), whereas it cannot be measured in the same way in terms of Polar coordinate (r, ϑ). This demonstrates the importance of the angle- preserving property of conformal mappings. For example, a conformal mapping $w = f(z)$ can be used to transform Laplace's equation from a complex domain D into simpler domain, such as the unit disk or the upper-half plane where the problem becomes easier to solve. The solution then is pulled back to D using the inverse mapping $z = f^{-1}(w)$. Consider the Dirichlet problem for Laplace's equation in a simply connected but irregular domain $D \subset \mathbb{C}$:

$z \in D, w \in \mathbb{D} = \{w \in \mathbb{C}, Im(w) > 0\}$; $\mathbb{D} = f(D)$

$$\nabla^2\psi(z) = 0 \text{ in } D, \quad \text{with } \psi(z) = \varphi(z) \text{ on } \quad \partial D$$

Where $\varphi(z)$ is a given boundary function. Solving this problem directly in D is difficult due to the complex geometry of the domain. To simplify the problem, a conformal mapping is applied to map D onto $\mathbb{D}$.Under this transformation, the problem becomes:

$$\nabla^2\psi^*(w) = 0 \text{ in } \mathbb{D}, \text{ with } \psi^*(w) = \varphi^*(w) \text{ on } \partial\mathbb{D}$$

Where $\varphi^*(w) = \varphi(f^{-1}(w))$ is the transformed boundary function. Since the geometry of the upper -half plane is simple, the solution $\psi^*(w)$ can be obtained by Fourier series or Poisson's integral formula. See [18, p .45], [17, Sec. 1.3, p. 212]. Once $\psi^*(w)$ is found, $\psi(z)$ is recovered by composing with the conformal map. This approach is illustrated through an example in Section 9.

Note: Throughout this work, the terms *complex plane*, $z - plane$, and $\mathbb{C} - plane$ will be used interchangeably to denote the same concept.

1. Methodology

The methodology in this study consists of two main parts. First, the theoretical framework of conformal mappings is established through a review of harmonic and analytic functions, the Cauchy-Riemann equations, and the definition of conformal mappings as angle-preserving transformations with non-vanishing derivative. Basic examples such as linear maps, translation, rotation, and power mappings are included to illustrate how these mappings preserve angles and simplify geometric properties. A power mapping is first used to map the half-disk onto the full unit disk, followed by a Möbius transformation that sends this disk to the upper-half plane.

Second, in this domain, Poisson's formula provides the harmonic solution explicitly. Subsequently, the inverse mapping is applied to transfer the solution back to the original domain ensuring that the boundary conditions of the problem are satisfied.

2. Harmonic Function

2.1 Definition: In the $xy - plane$, a real valued function $u(x, y)$ is called harmonic on a region of the $xy - plane$ if its second-order partial derivatives u_{xx} and u_{yy} are continuous functions in that region, and satisfy **Laplace's condition** ($\nabla^2 u = 0$)

$$u_{xx} + u_{yy} = 0$$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Where, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the **Laplace Operator.**

Example 2.1.1

Consider the function $u(x, y) = \ln\sqrt{x^2 + y^2}$. It is defined everywhere in the plane except at the origin

By using the properties of logarithms, we can write

$$u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$$

$$\because u_x = \frac{2x}{2(x^2+y^2)} = \frac{x}{x^2+y^2} \Longrightarrow u_{xx} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\because u_y = \frac{2y}{2(x^2+y^2)} = \frac{y}{x^2+y^2} \Longrightarrow u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore u_{xx} + u_{yy} = \frac{-x^2+y^2+x^2-y^2}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = 0 \;\forall\; (x, y) \neq (0,0)$$

Therefore, this function provides a typical example of a harmonic function valid on the entire plane except the point $(0,0)$.

Example 2.1.2

$$u(x, y) = e^{-y}\sin x$$

$$u_x = e^{-y}\cos x \Longrightarrow u_{xx} = -e^{-y}\sin x$$

$$u_y = -e^{-y}\sin\, x \;\Longrightarrow u_{yy} = e^{-y}\sin x$$

$\therefore u_{xx} + u_{yy} = 0$ for all (x, y) in $xy - plane$.

3. Complex function

3.1Definition: Complex-valued functions of a single complex variable are the main focus of this study. A complex variable z is composed of a real part, denoted as $Re(z)$, and an imaginary part, denoted as $Im(z)$, where $i = \sqrt{-1}$ is the imaginary unit. A function $f(z)$, when defined on a set D, established a rule that maps each complex number $z \in D$ to a complex number w in a set D^*. The value of the function at z is w, written as $w = f(z)$. A complex function can be uniquely expressed as a complex combination of two real functions $u(x, y)$ and $v(x, y)$,i.e., $f(z) = u(x, y) + iv(x, y)$ .Where, $u = Re(f)$ and $v = Im(f)$. [11, p.277] and [3, p.2]

Any complex-valued function f of the complex variable z maps points in the $xy - plane$ to points in the $uv - plane$ via $w = f(z)$.

4. Complex differentiation

4.1 Definition of Complex Derivative

Differentiation in $\mathbb{C}-$ plane follows the same principle as in real analysis. $f(z)$ is differentiable at z_0 only if the limit used to define its derivative exists at that point

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \quad \ldots\ldots\ldots \textbf{(4.1)}$$

Then,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

Let $\Delta z = z - z_0$, then (4.1) can be rewritten as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(\Delta z + z_0)-f(z_0)}{\Delta z}$$

Δz denotes the change in the value of z. It must be noticed that f is a differentiable function on a set S iff it is differentiable at every single point z within that domain. Thus

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(\Delta z + z)-f(z)}{\Delta z}$$

If $w = f(z)$, then Δw signifies a change in the value of w corresponding to the change of z .

Let $\Delta w = f(\Delta z + z) - f(z)$, then

$$\frac{df}{dz} = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

4.2 Derivative of a function $w(t)$

To introduce the linear integral in complex analysis, it is important to define the derivative of complex-valued function w of a real variable τ. Let $w(\tau) = u(\tau) + iv(\tau)$, $\tau \in \mathbb{R}$. Then

$$w'(\tau) = \frac{dw}{dt} = u'(\tau) + iv'(\tau)$$

Where the two real- valued functions of t u *and* v are differentiable at τ , implies u' *and* v' exist at τ. See [1, p. 155]

4.3 Theorem: If a complex function f is differentiable at z_0, then this function is continuous at this point.

It is important to note that the converse of this theorem is not necessarily true. See (sec .7, Ex .6,7)

5. Analytic function

5.1 Definition: A complex function $f: \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic function if it can be represented locally by a convergent power series. This means that the Taylor series for $f(z)$ at any point z^* within its domain is convergent.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z^*)(z-z^*)^n}{n!} \quad ; \ |z - z^*| < r$$

This series converges to $f(z)$ within the disk $|z - z^*| < r$. r is called the *Radius of Convergence* and it represents the distance to the nearest singularity of this function. $f^{(n)}(z^*)$ is the n-the derivative of the function evaluated at the point z^*.

5.2 Definition: A complex function $w = f(z) = u + iv$ is an analytic function in a region R if and only if its real and imaginary parts are continuously differentiable and satisfy the Cauchy-Riemann equations in R. In other words, u_x, u_y, v_x and v_y are continuous functions and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Example 5.2.1

Take $w = f(z) = z^2 + 1$. Writing it in terms of x and y gives $f(z) = (x^2 - y^2 + 1) + i2xy$. Thus,

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$

As it can be seen that the partial derivatives of its real and imaginary parts are continuous and satisfy Cauchy-Riemann conditions everywhere, which shows that this polynomial function is analytic across the entire complex plane.

Example 5.2.2: For the conjugate function $f(z) = \bar{z} = x - iy$, the Cauchy-Riemann condition $u_x = v_y$ is not satisfied, therefore this function **is not** analytic

$$u_x = 1, v_y = -1 \Rightarrow u_x \neq v_y$$

Example 5.2.3

Let $f(z)$ be the fractional function $\frac{1}{|z|^2-1}$. Clearly, this function is analytic inside and outside the unit circle $|z| = 1(\forall z \in \mathbb{C}; |z| \neq 1)$. [1, p. 72]

5.4 Relation between Harmonic and Analytic Function

If $f(z) = w = u(x,y) + iv(x,y)$ said to be analytic function, then the real part of this function is harmonic. This means that every harmonic function is the real part of an analytic function. [18, Theorem 4.1]

5.5 Difference Between Analytic and Differentiable

Analytic function is differentiable, however not every differentiable function is analytic. As introduced in [1, p.72], if $f(z)$ is differentiable at all points in an open set S, $i.e, f'(z)$ exists $\forall z \in S$ then the function is said to be analytic in S.In fact, analytic function has wider and stronger meaning in complex plane compared to differentiable function.

6. Curves and Angles in the Complex Plane

6.1 Curves (Arcs)

A set of all points $\mathcal{Z} = (x, y)$ in the complex plane $\mathbb{C}$ is called an $\boldsymbol{arc}$, denoted as C, if it can be represented by the parametric equations

$$x = x(\tau), y = y(\tau) \; ; \; a \leq \tau \leq b$$

Where, $x(\tau)$ and $y(\tau)$ are real continuous functions, C is parametrized by this complex-valued $\mathcal{Z}(\tau)$ where τ is called the real parameter. A curve C actually represent a continuous mapping from a real interval $[a,b] \subset \mathbb{R}$ to $\mathbb{C}$. $C = \{z: z = \mathcal{Z}(t), a \leq \tau \leq b\}$

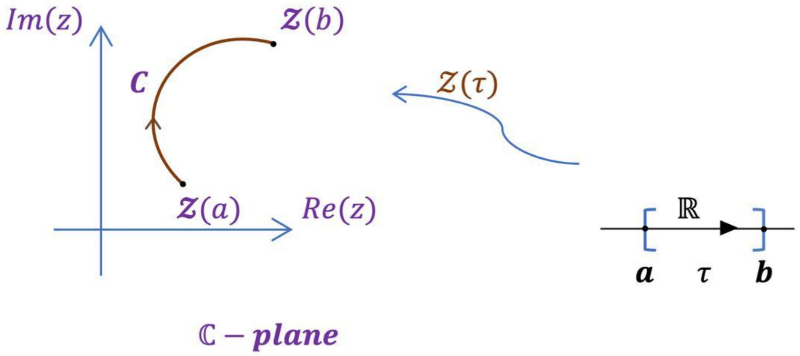

Figure 6.1: A complex function mapping a real interval to a curve C in the complex plane.

6.2 Tangent vector

The tangent vector of a curve C in $\mathbb{C}-$ plane is a vector that is tangent to the curve C at a given point. If a curve C is defined by the function $\mathcal{Z}(\tau) = x(\tau) + iy(\tau)$ over the interval $[a, b]$, its tangent vector is given by the derivative

$$v = \mathcal{Z}'(\tau) = (x'(\tau), y'(\tau))$$

This vector v indicates the direction in which C is moving at that specific point.

6.3 Angle between two curves

Consider two smooth curves, C_1 *and* C_2 in a domain D of the complex z-*plane* that intersect at a point $\mathcal{Z}_0$. Let $\mathcal{Z}_1(\tau)$ *and* $\mathcal{Z}_2(\tau)$ be their parametrizations of respectively such that they intersect when $\tau = \tau_0$.i.e.,

$$\mathcal{Z}_0 = \mathcal{Z}_1(\tau_0) = \mathcal{Z}_2(\tau_0)$$

The angle ϑ between these two curves is defined as the angle between their respective tangent vectors at their intersection point. To find this angle, we first calculate the tangent vectors v_1 and v_2 at τ_0:

$$v_1 = \mathcal{Z}_1{}' = x'_1(\tau_0) + iy'_1(\tau_0)$$

$$v_2 = \mathcal{Z}_2{}' = x'_2(\tau_0) + iy'_2(\tau_0)$$

Then,

$$\vartheta = \vartheta_2 - \vartheta_1 = Arg(v_2) - Arg(v_1)$$

$$= Arg(\frac{v_2}{v_1})$$

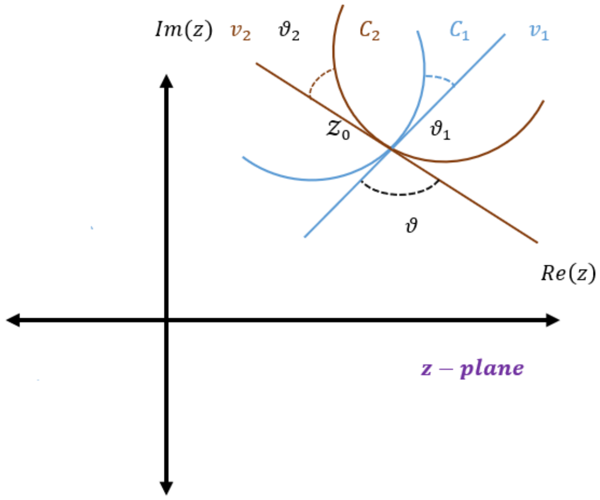

Figure 6.2: The angle between two curves C_1 *and* C_2

7. Complex function as mappings

7.1 Conformal mapping

Suppose that $w = f(z) = u(x, y) + iv(x, y)$ where $u(x, y)$ *and* $v(x, y)$ are real-valued functions is a transformation that maps the point $z = (x, y)$ in the complex $z - plane$ into the point $w = (u, v)$ in the complex $w - plane$. Then, this transformation is called **Conformal Mapping** if it preserves the angles. In other words, let C_1 *and* C_2 be two smooth arcs in $z - plane$ with intersection point $p_0(x_0, y_0)$. These two curves are mapped into C_1^* *and* C_2^* in $w - plane$ at the intersection point $p^*(u_0, v_0)$ under $w = f(z)$. If the angle between C_1 *and* C_2 at p_0 in $z - plane$ is equal both in magnitude and sense (clockwise or counter-clockwise) to the angle between C_1^* *and* C_2^* at w_0 in $w - plane$, then this transformation is known as Conformal Mapping.

7.2 Theorem

If w=$f(z)$ is an analytical function and its derivative $f'(z)$ is not zero, then f is a conformal mapping.

Proof:

Let $w = f(z)$ be analytic and $f'(z) \neq 0$ for all z in a domain D and consider the two smooth curves C_1 and C_2 with an intersection point z_0 in $z - plane$. Let $v_1 and$ v_2 be their tangent vectors respectively. The angle between the two curves is defined as

$$\vartheta = \arg(v_2) - \arg(v_1)$$

The two curves $C_1 and$ C_2 in $z - plane$ are mapped onto the two curves $C_1{}'$ and C'_2 in $w - plane$ respectively under the mapping $w = f(z)$. In addition, the image of $v_1 and$ v_2 are $f'(z_0)v_1 and$ $f'(z_0)v_2$ respectively in $w - plane$ and as a result the angle between the transformed vectors is defined as

$$\vartheta' = \arg(f'(z_0)v_2) - \arg(f'(z_0)v_1)$$

Using the properties of complex multiplication as following:

$$\vartheta' = \arg(f'(z_0)) + \arg(v_2) - [\arg(f'(z_0)) + \arg(v_1)]$$

$$=\arg(v_2) - \arg(v_1)$$

$\Rightarrow \vartheta = \vartheta'$. This proves that $f(z)$ preserves angles.

Now, since $w = f(z) = u + iv$ is analytic function, it must satisfy the Cauchy -Riemann equations

$u_x = v_y$ and $u_y = -v_x$. The Jacobian of $f(z)$ is

$$J_f = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Applying the Cauchy -Riemann equations, the determinant of J_f is

$$\det(J_f) = u^2{}_x + v^2{}_x = |f'(z)|^2$$

Because $|f'(z)|^2 \neq 0$ implies $\det(J_f) > 0$ and as a result of that $f(z)$ is invertible and orientation-preserving.

Example 7.2.1: The mapping $w = f(z) = z^2$ sends the first quadrant in $z - plane$ onto the upper half of the image plane. Since its derivative $f'(z) = 2z$ vanishes only at $z = 0$, the transformation is conformal throughout the plane except at the original.

7.3 Remark: The converse of Theorem 7.2 also holds in reverse: Any planar conformal map is generated by a complex analytic function with non-zero derivative. see [3, p. 35]

In addition, $w = f(z)$ is called an **Isogonal** mapping if it preserves the magnitude of angles, but not their orientation. See [1, p. 347]

Example 7.3.1: let $f(z) = \bar{z}$. As it can be shown, this transformation is a reflection in the real axis. $f(z)$ here is isogonal but not conformal. According to theorem 7.2, the function is not conformal because it is not analytic, as it does not satisfy the Cauchy-Riemann equations.

8. Special Transformations

8.1 Translation with *b* units.

It is given by the form $w_1 = f(z) = z + b$ where $b = b_1 + ib_2$ nonzero complex constant. w_1 here translates the entire plane by b, shifiting z horizontally by b_1 and vertically by b_2.

8.2 Rotations

Let $w_2 = f(z) = az$ where a is a complex constant.

$let\ a = \beta e^{i\varphi}; \beta = |a|$, $\quad -\pi < \varphi \leq \pi$

$z = re^{i\theta}$ $\forall z \in \mathbb{C}$; $r = |z|, -\pi < \theta \leq \pi$

Then, $w_2 = f(z) = az = r\beta e^{i(\theta+\varphi)}$ is considered a rotation transformation, which rotates the vector $\vec{z} = (x, y)$ around the origin point with angle φ. Note that if $0 < \beta < 1$, then w_2 will increase r by β, otherwise, it decreases r by β.

8.3 Inversions

It has the general form $w_3 = T(z) = \frac{1}{z}$, $z \neq 0$

Here, $|w_3| = \frac{1}{|z|}$ and $\arg(w_3) = \arg(1) - \arg(z) = 0 - \arg(z) = -\arg(z)$. The two-line segments $|z| and\ |w_3|$ are collinear and then there are three cases

. If $|z| < 1$, then $|w_3| > 1$,i.e., every point in the unit circle $|z| = 1$ inverse to a point outside this circle
. If $|z| > 1$, then $|w_3| < 1$,i.e., every point outside the unit circle $|z| = 1$ is inverse to a point inside this circle.
. If $|z| = 1$, then $|w_3| = 1$. In other words, the points are mapped to themselves.

8.4 Mobius transformation

It is a fractional linear transformation (rational function) of the form

$$f(z) = \frac{az+b}{cz+d} \text{ , } ad - bc \neq 0 \text{……… } \textbf{(8.1)}$$

Where $a, b, c\ and\ d \in \mathbb{C}$.This transformation is considered as a fundamental class of conformal mapping in complex analysis. Which map the extended complex plane $\mathbb{C} \cup \{\infty\}$ onto itself and preserve angles, lines and circles.

As it can be seen, the Möbius transformation (8.1) is defined as a sequence of these three linear fractional transformations:

$$z \xrightarrow{(translation)} z + \frac{d}{c} \xrightarrow{(inversion)} \frac{1}{z+\frac{d}{c}} \xrightarrow{(rotation-magnifigation)} \left(\frac{bc-ad}{c^2}\right)\left(\frac{1}{z+\frac{d}{c}}\right)$$

$$\xrightarrow{translation} \left(\frac{bc-ad}{c^2}\right)\left(\frac{1}{z+\frac{d}{c}}\right) + \frac{a}{c} = \frac{az+b}{cz+d}$$

9. Applications

Consider the Dirichlet boundary-value problem defined on the upper half of the unit disk $|z| < 1$.

Let D be the upper half of the unit disk, $D = \{z \in \mathbb{C}, |z| < 1, Im(z) > 0\}$. Solve Laplace's equation $\nabla^2\psi(z) = 0$ in D with boundary conditions (∂D)

- $\psi(z) = 0$ on the upper semi-circle $|z| = 1$, $Im(z) \geq 0$
- $\psi(z) = \alpha\ \forall\ |Re(z)| < 1$, $Im(z) = 0$

The aim is to map the upper half disk D to the upper half plane, where we can use Poisson's formula.
Since Möbius transformation is a conformal mapping, which preserves angles, a single Möbius transformation cannot map a half-disk onto a full disk. To solve this problem, a composition of conformal mappings is need to be used.

Steps to solve using conformal mapping:

1- The mapping $w_1 = z^2$ sends the half disk D onto the entire unit disk $|z| = 1$. This step simplifies the geometry while preserving the boundary conditions. As it can be verified, that every point in D has angle ϑ with $0 \leq \vartheta \leq \pi$. w_1 maps ϑ to 2ϑ with $0 \leq 2\vartheta \leq 2\pi$.Thus the full unit disk is covered.

2- Applying the Möbius transformation $w_2 = i\frac{1-z}{1+z}$. This conformally transfers the unit disk onto the upper half -plane. Thus, the composition

$$w = w_2(w_1) = i\frac{1-z^2}{1+z^2}$$

provides a conformal mapping from the half -disk D to the half-plane $Im(w) > 0$.

3- The new domain now is the upper half-plane, $Im(w) > 0$. The original boundary conditions are applied to the new, simplified domain:

- $\psi^*(u,0) = 0 \ \forall \ u > 0$
- $\psi^*(u,0) = \rho \ \forall \ u < 0$

These conditions form a simple step function on the real axis of the new domain.

4- The harmonic solution to this problem is simply the imaginary part of the function (see [17],[18])

$$k \text{ Log}(w) = k \ (\ln|w| + i \arg(w))$$

The constant k is determined by the boundary conditions and arg (w) is the argument(angle) of the complex number w. A more general solution for a step function boundary condition on the real axis is given by

$$\psi^*(u,v) = \frac{\rho}{\pi} \arg(w), \quad -\pi < \arg(w) \le \pi \quad \text{.......}\mathbf{(9.1)}$$

Verifying the proposed solution (9.1) to ensure it satisfies the given boundary conditions:

For the positive real axis $w = u + iv$

- $\forall \ w = u$ with $u > 0$, $\arg(w) = 0$. By substituting this into (9.1), we get $\psi^*(u,v) = k.0 = 0$
- $\forall \ w = u \ < 0$ with $u < 0$, $\arg(w) = \pi$. Substituting this into (9.1), we get $\psi^*(u,v) = k\pi$. This must match the boundary condition $\psi^*(u,0) = \rho$ on this part of the boundary. As a result of that, $k\pi = \rho$ which implies $k = \frac{\rho}{\pi}$ is the total potential difference(voltage)between one side of the wedge and the other

5- Finally, map the solution back to the original domain by substituting the conformal mapping $w = i\frac{1-z^2}{1+z^2}$ back into the solution $\psi^*(u,v)$ to find the solution $\psi(x,y)$ in the original domain:

$$\psi(z) = \frac{\rho}{\pi} \arg\left(i\frac{1-z^2}{1+z^2}\right)$$

And this is the solution to the original Dirichlet problem, where the function $\psi(z)$ satisfies the prescribed boundary conditions on D. This result works for any boundary potential difference, and we can plug in a specific value, such as $\rho = 1$) if needed.

Remark:

This solution is equivalent to applying the *Poisson integral formula* in the upper half-plane. In fact, the solution to the harmonic problem in the upper-half plane with step function boundary conditions is found by using Poisson's integral formula for a half-plane

$$\psi^*(u,v) = \frac{v}{\pi}\int_{-\infty}^{\infty} \frac{\psi(t)}{(u-t)^2 + v^2} dt$$

Given the boundary conditions $\psi(t) = 0$ for $t > 0$ and $\psi(t) = \rho$ for $t < 0$, the integral becomes:

$$\psi^*(u,v) = \frac{v}{\pi}\int_{-\infty}^{0} \frac{\rho}{(u-t)^2 + v^2} dt$$

Which evaluates to

$$\psi^*(u,v) = \frac{\rho}{\pi}\tan^{-1}\left(\frac{u}{v}\right) + \frac{\rho}{2} = \frac{\rho}{\pi}\arg(w)$$

This demonstrates how the solution to the *Dirichlet problem* in the half-plane is derived directly from the Poisson integral formula, and that it is equivalent to the imaginary part of the analytic function $\frac{\rho}{\pi}\text{Log}(w)$for this specific.

Conclusion

This paper has highlighted the significance of conformal mappings in complex analysis, particularly their ability to preserve angles and local geometric structures. Such mappings simplify the process of solving boundary value problems by reducing them to equivalent problems in more manageable domains. The paper demonstrated this concept through the application of solving Laplace's equation in the complex domain. It showed how conformal mapping can transform the problem from an irregular domain to a unit disk, where it can be solved easily. This example emphasises how conformal mappings can be a powerful and effective tool for simplifying problems in complex analysis while maintaining both mathematical accuracy and rigor.

Extending this work, future studies could focus on developing new computer algorithms and simulation models that apply conformal mappings to effectively solve complex problems in areas, such as fluid dynamics and aerodynamics, where the angle-preserving property of these mappings supports physical fidelity.

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