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On the Uniqueness and Existence of Solutions for Linear and Nonlinear Impulsive Second-Order Differential Equations with Applications to Neural Oscillators

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الوحدانية والوجود لحلول المعادلات التفاضلية الاندفاعية من الرتبة الثانية الخطية وغير الخطية مع تطبيقات على المذبذبات العصبية

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Abstract:

This paper investigates the uniqueness and existence of solutions for second-order differential equations from both classical and modern perspectives. In the first part, classical techniques such as energy functions and the Grönwall–Bellman inequality are employed to re-establish the uniqueness of solutions for second-order linear homogeneous equations with continuous coefficients. In the second part, the analysis is extended to nonlinear impulsive systems inspired by biological neural oscillators and Central Pattern Generators (CPGs), which include time-dependent damping and instantaneous state changes. Existence and uniqueness of periodic or almost-periodic solutions are established using topological fixed-point theory and Lyapunov-like functions under biologically realistic assumptions. Numerical simulations using MATLAB are presented to validate the theoretical findings. The results have direct applications in bioengineering, neural dynamics, and control systems where hybrid continuous-discrete behavior is present.

Keywords: Uniqueness of solutions, Grönwall–Bellman inequality, Impulsive differential equations, Neural oscillators, Periodic solutions, Lyapunov functions, Central Pattern Generators (CPGs).

الملخص
تبحث هذه الورقة في الوجود والوحدانية للحلول للمعادلات التفاضلية من الرتبة الثانية من منظورين، الكلاسيكي والحديث. في الجزء الأول، يتم إعادة توظيف بعض الأدوات الكلاسيكية مثل دوال الطاقة و متباينة غرونوول–بيلمان لإثبات الوحدانية من جديد للمعادلات الخطية المتجانسة ذات المعاملات المستمرة. أما في الجزء الثاني، فيتم توسيع نطاق التحليل ليشمل الأنظمة غير الخطية الاندفاعية المستوحاة من المذبذبات العصبية و مولدات الأنماط المركزية (CPGs)، والتي تتضمن تخميدًا يعتمد على الزمن وتغيرات آنية في الحالة. وبافتراضات واقعية بيولوجيًا، يتم إثبات وجود ووحدانية الحلول الدورية وشبه الدورية باستخدام نظرية النقاط الثابتة الطوبولوجية مع دوال شبيهة بليابونوف. ولتعزيز النتائج النظرية، يتم عرض محاكاة عددية باستخدام برنامج MATLAB توضّح الديناميكيات الهجينة (المستمرة–المنفصلة) للمذبذبات العصبية. وتوفر هذه النتائج إطارًا رياضيًا صارمًا مع تطبيقات مباشرة في الهندسة الحيوية، وديناميكيات الأعصاب، وأنظمة التحكم حيث يظهر السلوك الاندفاعي والهجين بشكل جوهري.

الكلمات المفتاحية: الوجود والوحدانية للحلول؛ متباينة غرونوول –بيلمان؛ المعادلات التفاضلية الاندفاعية؛ المذبذبات العصبية؛ الحلول الدورية وشبه الدورية؛ دوال ليابونوف؛ مولدات الأنماط المركزية (CPGs).

Introduction

Differential equations are fundamental tools in modeling the dynamics of physical, biological, and engineering systems. In particular, second-order differential equations play a critical role in describing systems with acceleration, such as mechanical vibrations, electrical circuits, and neural oscillations. The uniqueness of solutions to these equations is essential for ensuring the predictability and stability of the modeled system.

Classical results on the uniqueness of solutions to second-order linear differential equations rely on standard theorems under continuity and Lipschitz conditions. Notable methods include Ziebur's theorem [1], which offers uniqueness for linear homogeneous equations under continuous coefficients, and the Grönwall–Bellman inequality [2], widely used to derive bounds and prove uniqueness in more general settings. These techniques form the backbone of the linear theory.

However, real-world systems often exhibit abrupt changes or discontinuities due to environmental shocks, control inputs, or biological signals. This motivates the study of impulsive differential equations (IDEs), where the state or its derivative experiences instantaneous jumps at specified moments. IDEs have been extensively studied in the context of control theory [3], biological systems [4], and neural models, such as central pattern generators (CPGs) [5].

Recent studies have focused on the existence and stability of solutions to IDEs, especially under nonlinear and time-varying conditions [6]. The use of topological fixed-point theory, along with Lyapunov-like functions, has proven effective in analyzing periodic and almost-periodic solutions in such systems [7]. Despite these advances, there remains a gap in unifying classical linear theory with nonlinear impulsive frameworks, especially from the perspective of both theoretical proof and numerical validation.

The aim of this paper is twofold. First, we revisit and re-establish classical uniqueness results for second-order linear differential equations using energy function methods and integral inequalities. Second, we extend the analysis to nonlinear impulsive systems with biologically motivated impulses, and prove the existence and uniqueness of solutions under realistic assumptions using fixed-point methods. The theoretical work is supplemented with MATLAB simulations to validate both linear and impulsive cases, highlighting applications in mechanical systems (mass-spring-damper) and neural dynamics (bursting oscillators and CPGs).

The novelty of this work lies in bridging classical differential equation theory with modern impulsive and biologically inspired nonlinear systems. Unlike most existing works that treat these areas separately, this paper provides a unified treatment, offering both theoretical uniqueness proofs and practical simulations. In particular, we extend Ziebur's linear uniqueness framework to cover impulsive differential equations with time-dependent damping and state-dependent impulses, motivated by neural and mechanical models.

Second-order linear differential equations are foundational in mathematical modeling across physics, engineering, and biology. A classic form involves the equation:

$$\left.\begin{array}{c} u''(t) + g(t)u'(t) + h(t)u(t) = 0 \\ u(t_0) = u_0, u'(t_o) = u_1 \end{array}\right\}$$

Where $g(t)$, $h(t)$ are continuous. This is a second-order linear homogeneous ODE with initial conditions. The goal is to prove that it has only one solution.

under appropriate initial conditions. The existence and uniqueness of solutions to such initial value problems (IVPs) are crucial for the predictability of models (Coddington & Levinson, 1955). Ziebur's theorem provides one such uniqueness result under the assumption that the coefficients $g(t)$ and $h(t)$ are continuous on a closed interval (Ziebur, 1968). More recently, generalized inequalities such as Grönwall's lemma have played a key role in proving uniqueness results for nonlinear and time-dependent systems (Walter, 1998; Agarwal, 1993).

Uniqueness in Linear Systems

Theorem (A–D–Z–Ziebur):

Let $g(t)$ and $h(t)$ be continuous on the $J = [t_0, t_0 + c]$, where $c > 0$. Then the initial value problem (IVP):

$$u''(t) + g(t)u'(t) + h(t)u(t) = 0$$

$$u(t_0) = u_0, u'(t_o) = u_1$$

has a unique solution on the interval J.

Step-by-step Proof:

Step 1: Assume two solutions

Let $u(t)$ and $v(t)$ be two solutions to the IVP. Since the equation is linear and homogeneous, define:

$$w(t) = u(t) - v(t)$$

Then:

- $w(t_0) = u(t_0) - v(t_0) = 0$
- $w'(t_0) = u'(t_0) - v'(t_0) = 0$

Step 2: Write the equation for w(t)w(t)w(t)

Since both $u(t)$ and $v(t)$ satisfy the same differential equation:

$$u''(t) + g(t)u'(t) + h(t)u(t) = 0$$
$$v''(t) + g(t)v'(t) + h(t)v(t) = 0$$

Subtracting, we get:

$$w''(t) + g(t)w'(t) + h(t)w(t) = 0$$

So $w(t)$ satisfies the **same homogeneous linear equation** with **zero initial conditions**.

Step 3: Define an energy-like function

Let:

$Z(t) = w^2(t) + [w'(t)]^2$

This is always ≥ 0, and represents a kind of "energy" in the system.

Step 4: Compute the derivative of $Z(t)$

$$Z'(t) = 2w(t)w'(t) + 2w'(t)w''(t)$$

Now use the differential equation:

$$w''(t) = -g(t)w'(t) - h(t)w(t)$$

Substitute into the derivative:

$$Z'(t) = 2w(t)w'(t) + 2w'(t)[-g(t)w'(t) - h(t)w(t)]$$
$$= 2w(t)w'(t) - 2g(t)[w'(t)]^2 - 2h(t)w(t)w'(t)$$

Group terms:

$Z'(t) = 2w(t)w'(t)\big(1 - h(t)\big) - 2g(t)[w'(t)]^2$

To estimate this, use absolute values:

$$|2w(t)w'(t)| \leq w^2(t) + [w'(t)]^2 = Z(t)$$

(by AM GM inequality)

And we know $g(t)$ and $h(t)$ are continuous ⇒ bounded on J.

So there exists a constant $C > 0$ such that:

$$Z'(t) \leq C \cdot Z(t)$$

Step 5: Apply Grönwall's Inequality

We have:

$$Z'(t) \leq C \cdot Z(t),\ Z(t_0) = 0$$

By Grönwall's inequality:

$$Z(t) \leq Z(t_0)e^{C(t-t_0)} = 0$$

Thus, for all $t \in J$:

$Z(t) = 0 \Rightarrow w(t) = 0$ and $w'(t) = 0$

Then:

$$w(t) = u(t) - v(t) = 0 \Rightarrow u(t) = v(t)$$

Hence, the solution is **unique** on the interval J.

3. Application: Mass-Spring-Damper System

This is a classic physical system modeled by a second-order linear ODE of the form:

$$mu''(t) + cu'(t) + ku(t) = 0$$

Which we can rewrite as:

$$u''(t) + \frac{c}{m}u'(t) + \frac{k}{m}u(t) = 0$$

This exactly matches the form:

$$u''(t) + g(t)u'(t) + h(t)u(t) = 0$$

Assume $g(t) = \frac{c}{m}$, and $h(t) = \frac{k}{m}$.

Both constants ⇒ continuous ⇒ theorem applies.

3.1 Solve this IVP using MATLAB numerically.

MATLAB Code

Here's a simple MATLAB script to simulate the system using ode45, a built-in solver for ODEs:


```
% Mass-Spring-Damper System Simulation
% u'' + (c/m) u' + (k/m) u = 0
% Parameters
m = 1;        % Mass (kg)
c = 0.5;      % Damping coefficient
k = 4;        % Spring constant

% Rewrite as a system:
% u1 = u, u2 = u'
% Then: u1' = u2, u2' = -(c/m)*u2 - (k/m)*u1

% Define system
g = @(t, u) [u(2); - (c/m)*u(2) - (k/m)*u(1)];

% Time span
tspan = [0 10];

% Initial conditions: u(0) = 1 (displacement), u'(0) = 0 (velocity)
u0 = [1; 0];

% Solve ODE
[t, U] = ode45(g, tspan, u0);

% Plot results
figure;
plot(t, U(:,1), 'b-', 'LineWidth', 2);
xlabel('Time (s)');
ylabel('Displacement u(t)');
title('Mass-Spring-Damper System Response');
grid on;
```

After run MATLAB, we obtain the following Figure 1

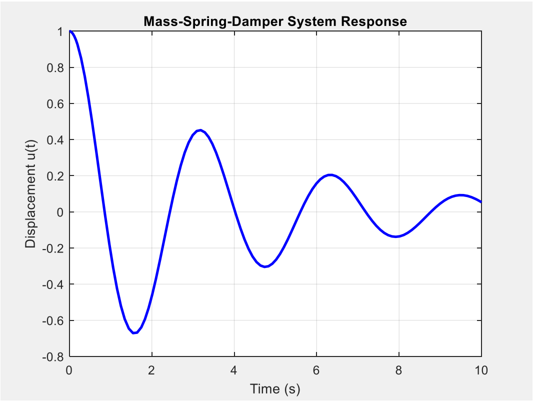

Figure 1: Mass-Spring-Damper System Response .

3.2 Interpretation of the Output:

- This models a damped harmonic oscillator.
- The plot shows how the system returns to equilibrium over time, depending on damping.
- Existence and uniqueness ensure that this numerical solution approximates **one and only one** true solution.
-

4. Existence and Uniqueness of Periodic Solutions in Nonlinear Neural Oscillators with Impulsive Dynamics and Time-Dependent Damping.

- **Impulsive effects** (sudden jumps in state due to synaptic firing),
- **Time-dependent or state-dependent damping**,

- **Fuzzy or uncertain nonlinearities**,
- And even **neutrosophic logic components** in high-uncertainty environments.

To guarantee the existence and uniqueness of a solution (whether for simulation or real-time control), we rely on theorems like the one you quoted—ensuring that the system's behavior is mathematically sound and well-defined.

This study addresses the existence and uniqueness of solutions for a class of impulsive second-order nonlinear differential equations motivated by neural oscillators, particularly bursting neurons and Central Pattern Generators (CPGs). Extending the classical Ziebur Existence Theorem, the framework incorporates state-dependent damping and discrete impulses, capturing abrupt synaptic events or control actions in hybrid dynamical systems.

4.1 Mathematical Model

We consider the general form:

$$u''(t) + g\big(t, u(t), u'(t)\big) + h\big(t, u(t)\big) = 0, t \neq t_k,$$

with impulsive conditions at discrete times $t = t_k$:

$\Delta u \,|_{t=t_k} = I_k\big(u(t_k^-)\big), \Delta u' \,|_{t=t_k} = J_k\big(u'(t_k^-)\big)$

where $\Delta u = u(t_k^+) - u(t_k^-)$ represents the instantaneous jump due to an impulse.

- The function $g(t, u, u')$ represents a **nonlinear damping force**.
- The function $h(t, u)$ represents a **nonlinear restoring force**.
- The mappings I_k and J_k describe **impulsive effects**, which may model phenomena such as **synaptic bursts** in biological systems or **external control inputs** in physical systems.
-

To establish the existence and uniqueness of **periodic** or **almost-periodic solutions**, we employ **topological fixed-point theory** in combination with **Lyapunov-like functions**, under biologically and physically realistic assumptions.

4.2 Applications

We are working with an **impulsive second-order nonlinear differential equation**:

$$u''(t) + g\big(t, u(t), u'(t)\big) + h\big(t, u(t)\big) = 0, t \neq t_k,$$

With impulsive conditions:

$$\Delta u \,|_{t=t_k} = I_k\big(u(t_k^-)\big), \Delta u' \,|_{t=t_k} = J_k\big(u'(t_k^-)\big)$$

We will now create a simplified numerical example with:

- Linear damping: $g(t, u, u') = \alpha u'$
- Linear restoring: $h(t, u) = \beta u$
- Impulses:
  - $I_k\big(u(t_k^-)\big) = A \cdot u$
  - $J_k\big(u'(t_k^-)\big) = B \cdot u'$

This gives:

$$u''(t) + \alpha u'(t) + \beta u(t) = 0, t \neq t_k$$

$u(t_k^+) = u(t_k^-) + Au(t_k^-), u'(t_k^+) = u'(t_k^-) + Bu'(t_k^-)$

4.3 Applications and Case Studies

MATLAB Code (Simplified Example)

```
% Impulsive second-order differential equation solver

% Parameters
alpha = 0.5;      % Damping coefficient
beta = 2;         % Restoring coefficient
A = 0.2;          % Impulse magnitude for u
B = -0.3;         % Impulse magnitude for u'
T = 20;           % Total simulation time
dt = 0.01;        % Time step
tk = 5:5:T-1;     % Impulse times

% Initialization
t = 0:dt:T;
u = zeros(size(t));
```



```
v = zeros(size(t));  % v = u'
u(1) = 1;            % Initial condition
v(1) = 0;

% Time stepping
for i = 2:length(t)
   % Euler method
   u(i) = u(i-1) + dt * v(i-1);
   v(i) = v(i-1) + dt * (-alpha * v(i-1) - beta * u(i-1));

   % Check for impulses
   if any(abs(t(i) - tk) < dt/2)
      u(i) = u(i) + A * u(i);
      v(i) = v(i) + B * v(i);
   end
end

% Plotting
figure;
plot(t, u, 'b', 'LineWidth', 2); hold on;
plot(t, v, 'r--', 'LineWidth', 2);
xlabel('Time');
ylabel('u(t) and u''(t)');
legend('u(t)', 'u''(t)');
title('Impulsive Second-Order Differential Equation');
grid on;
```

We get that:

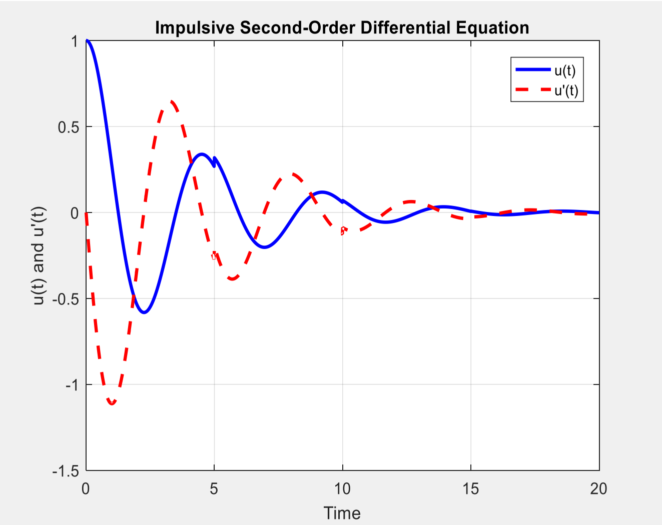

Figure 2: Impulsive Second-Order Differential Equation.

5 Lemma (Gronwall-Bellman Lemma for DE)

Let C≥ 0, v be positive continuous function, $u \in C'(I)$ and positive such that

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} u(s)v(s)ds \text{ for all } t \geq t_0$$

Then

$$u(t) \leq \mathrm{C}e^{\int_{t_0}^{t} v(s)ds}$$

Proof
Let

$$w(t) = C + \int_{t_0}^{t} u(s)v(s)ds$$

$u(t) \leq w(t)$ & $w(t_0) = C$
Then

$$w'(t) = u(t)v(t) \leq w(t)v(t) \Longrightarrow$$

$$\int_{t_0}^{t} \frac{dw(s)}{w(s)} \leq \int_{t_0}^{t} v(s)ds \Longrightarrow$$

$$Ln\left|\frac{w(t)}{w(t_0)}\right| \leq \int_{t_0}^{t} v(s)ds \Longrightarrow$$

$$\frac{w(t)}{w(t_0)} \leq e^{\int_{t_0}^{t} v(s)ds} \Longrightarrow$$

$$(t) \ \leq Ce^{\int_{t_0}^{t} v(s)ds} \Longrightarrow$$

$$u(t) \leq Ce^{\int_{t_0}^{t} v(s)ds} \qquad \text{for all } t \geq t_0$$

5.2 Lemma (Gronwall-Bellman Lemma for IDE)
Let $C \geq 0, B_i \geq 0$and $v \in PC'(I)$, $v > 0$ continuous function and $u \in PC'(I)$is positive and satisfies

$$u(t) \leq C + \int_{t_0}^{t} u(s)v(s)ds + \sum_{t_0 \leq \theta_i < t} \beta_i u(\theta_i) ----(5.1)$$

for all $t \geq t_0$

Then

$$u(t) \leq ce^{\int_{t_0}^{t} v(s)ds} \prod_{t_0 \leq \theta_i < t} (1 + \beta_i) \quad ------- \ (5.2)$$

Proof
We will use induction on the interval $(\theta_i, \theta_{i+1}]$, then when
$t \in [t_0, \theta_1]$, hence the inequality (5.1) becomes

$$u(t) \leq C + \int_{t_0}^{t} u(s)v(s)ds --------- \quad (5.3)$$

By using G.B Lemma, we have

$$u(t) \leq Ce^{\int_{t_0}^{t} v(s)ds} \quad \forall t \in [t_0, \theta_1] \qquad (5.4)$$

Which means that (2) satisfies for $t \in [t_0, \theta_1]$ by (5.4)
Let $t \in (\theta_1, \theta_2]$, then the inequality (5.1) becomes

$$u(t) \leq C + \int_{t_0}^{t} u(s)v(s)ds + B_1 u(\theta_1) ------ (5.5)$$

$$= C + \int_{t_0}^{\theta_1} u(s)v(s)ds + \int_{\theta_1}^{t} u(s)v(s)ds + B_1 u(\theta_1)$$

By using the Equation (5.4)

$$\leq C + \int_{t_0}^{\theta_1} u(s)v(s)ds + \int_{\theta_1}^{t} u(s)v(s)ds + B_1 Ce^{\int_{t_0}^{\theta_1} v(s)ds}$$

By using G.B Lemma, we have

$$\leq Ce^{\int_{t_0}^{\theta_1} v(s)ds} + \int_{\theta_1}^{t} u(s)v(s)ds + B_1 Ce^{\int_{t_0}^{\theta_1} v(s)ds}$$

$$= \mathrm{C}(1+\mathrm{B}_1)e^{\int_{t_0}^{\theta_1} v(s)ds} + \int_{\theta_1}^{t} u(s)v(s)ds$$

By using G.B Lemma, we have

$$\boldsymbol{u(t) \leq C(1+B_1)e^{\int_{t_0}^{\theta_1} v(s)ds} e^{\int_{\theta_1}^{t} v(s)ds}} \Rightarrow$$

$$\boldsymbol{u(t) \leq C(1+B_1)\, e^{\int_{t_0}^{t} v(s)ds}} \quad \forall \boldsymbol{t} \in (\boldsymbol{\theta_1}, \boldsymbol{\theta_2}] - - -(5.6)$$

Which means that (5.2) satisfies for $t \in (\theta_1, \theta_2]$ by (5.6)

Let $t \in (\theta_2, \theta_3]$, then the inequality (5.1) becomes as (5.7)

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} u(s)v(s)ds + B_1 u(\theta_1) + B_2 u(\theta_2) - -(5.7)$$

$$= C + \int_{t_0}^{\theta_2} u(s)v(s)ds + B_1 u(\theta_1) + \int_{\theta_2}^{t} u(s)v(s)ds + B_2 u(\theta_2)$$

By using the Equation (5.6)

$$u(t) \leq C(1+B_1)\, e^{\int_{t_0}^{\theta_2} v(s)ds} + \int_{\theta_2}^{t} u(s)v(s)ds + B_2 C(1+B_1)\, e^{\int_{t_0}^{\theta_2} v(s)ds}$$

$$= C(1+B_1)(1+B_2)e^{\int_{t_0}^{\theta_2} v(s)ds} + \int_{\theta_2}^{t} u(s)v(s)ds$$

By using G.B Lemma, we have

$$u(t) \leq C(1+B_1)(1+B_2)e^{\int_{t_0}^{\theta_2} v(s)ds} e^{\int_{\theta_2}^{t} v(s)ds} \Rightarrow$$

$$u(t) \leq C(1+B_1)(1+B_2)e^{\int_{t_0}^{t} v(s)ds} \; \forall t \in (\theta_2, \theta_3] - -(5.8)$$

The inequality(5.8) implies (5.2)

Assume that the inequality (5.1) is true for $t \in (\theta_k, \theta_{k+1}]$

That means

$$u(t) \leq \mathrm{C} + B_1 u(\theta_1) + B_2 u(\theta_2) + \cdots + B_k u(\theta_k) + \int_{t_0}^{t} u(s)v(s)ds - -(5.9)$$

implies

$$u(t) \leq c e^{\int_{t_0}^{t} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k}} (1+\beta_i) \; - - - - - (5.10)$$

Now, we need to prove that the inequality (5.1) satisfies (5.2) when $t \in (\theta_{k+1}, \theta_{k+2}]$

Then the inequality (5.1) becomes

$$u(t) \leq C + B_1 u(\theta_1) + B_2 u(\theta_2) + \cdots + B_k u(\theta_k) + B_{k+1} u(\theta_{k+1}) + \int_{t_0}^{t} u(s)v(s)ds \ldots (5.11)$$

$$= C + B_1 u(\theta_1) + B_2 u(\theta_2) + \cdots + B_k u(\theta_k) + \int_{t_0}^{\theta_{k+1}} u(s)v(s)ds + B_{k+1} u(\theta_{k+1}) + \int_{\theta_{k+1}}^{t} u(s)v(s)ds$$

By using the Equation (5.10), then (5.11) becomes

$$u(t) \leq c e^{\int_{t_0}^{\theta_{k+1}} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k}} (1+\beta_i) + B_{k+1} c e^{\int_{t_0}^{\theta_{k+1}} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k}} (1+\beta_i) + \int_{\theta_{k+1}}^{t} u(s)v(s)ds$$

$$= c e^{\int_{t_0}^{\theta_{k+1}} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k+1}} (1+\beta_i) + \int_{\theta_{k+1}}^{t} u(s)v(s)ds$$

By using G.B Lemma, we have

$$u(t) \leq C e^{\int_{t_0}^{\theta_{k+1}} v(s)ds} e^{\int_{\theta_{k+1}}^{t} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k+1}} (1+\beta_i) \Rightarrow$$

$$u(t) \leq Ce^{\int_{t_0}^{t} v(s)ds} \prod_{\substack{t_0 \leq \theta_i < t \\ i=1,2,\ldots,k+1}} (1+\beta_i) \; \forall t \in (\theta_{k+1}, \theta_{k+2}] \ldots \ldots \quad (5.12)$$

The inequality (5.12) implies (5.2)
Then the inequality (5.1) implies (5.2) for all $\forall t \in (\theta_{i+1}, \theta_{i+2}]$

5.3 Corollary 1

If $u \in PC(I)$ is a nonnegative function and C, ζ, L are nonnegative constants such that

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} [\zeta + Lu(s)]ds + \sum_{t_0 \leq \theta_i < t} [\zeta + Lu(\theta_i)] \ldots (5.13)$$

Then

$$u(t) \leq \left(C + \frac{\zeta}{L}\right) e^{L(t-t_0)} (1+L)^{i(t,t_0)} - \frac{\zeta}{L} \quad \ldots .. (5.14)$$

Where $i(t, t_0)$ is the number of θ_i in (t, t_0)
Proof
Let $\varphi(t) = \zeta + Lu(t) \Longrightarrow u(t) = \frac{\varphi(t)-\zeta}{L}$

$$u(t) \leq \mathrm{C} + \int_{t_0}^{t} \varphi(s)ds + \sum_{t_0 \leq \theta_i < t} \varphi(\theta_i) \Longrightarrow$$

$$\zeta + Lu(t) = \varphi(t) \leq \zeta + L\left[\mathrm{C} + \int_{t_0}^{t} \varphi(s)ds + \sum_{t_0 \leq \theta_i < t} \varphi(\theta_i)\right]$$

$$= \zeta + LC + \int_{t_0}^{t} L\varphi(s)ds + \sum_{t_0 \leq \theta_i < t} L\varphi(\theta_i)$$

By using G.B Lemma for IDE, We will have

$$\varphi(t) \leq (\zeta + LC)e^{\int_{t_0}^{t} Lds} \prod_{\substack{t_0 \leq k < t \\ k=1,2,\ldots,i}} (1+L) = (LC+\zeta)e^{L(t-t_0)}(1+L)^{i(t_0,t)}$$

Since $u(t) = \frac{\varphi(t)-\zeta}{L}$, then

$$\frac{\varphi(t)-\zeta}{L} \leq (\frac{\zeta}{L} + C)e^{L(t-t_0)}(1+L)^{i(t_0,t)} - \frac{\zeta}{L} \Longrightarrow$$

$$u(t) \leq \left(C + \frac{\zeta}{L}\right) e^{L(t-t_0)} (1+L)^{i(t_0,t)} - \frac{\zeta}{L}$$

Where $i(t, t_0)$ is the number of θ_i in (t, t_0)

5.4: Discussion

The results presented in this paper bridge a significant theoretical and practical gap between classical uniqueness theorems for linear differential equations and the emerging domain of nonlinear impulsive systems. The classical analysis, grounded in the Grönwall–Bellman inequality and Ziebur's theorem, reaffirms well-established uniqueness results under continuous coefficients. These foundational results are vital for applications in mechanical and electrical systems where smooth dynamics predominate.

In contrast, the extension to impulsive, nonlinear second-order systems addresses a class of problems far more aligned with contemporary modeling challenges in biological and hybrid dynamical systems. These systems are characterized by non-smooth behavior due to abrupt transitions—such as synaptic firing in neurons or external control impulses in engineered systems—which cannot be captured by traditional continuous models alone.

The incorporation of time-dependent damping and discrete impulses introduces analytical complexity but significantly enhances modeling fidelity, particularly for applications in Central Pattern Generators (CPGs) and bursting neural oscillators. The use of Lyapunov-like functions and topological fixed-point theory provides robust mathematical tools for proving existence and uniqueness of periodic or almost-periodic solutions, which are critical for biological realism and engineering control.

The numerical simulations complement the theoretical analysis, providing insight into the qualitative behavior of the systems under study. In particular, the MATLAB implementations illustrate the damped oscillatory behavior in classical systems and the hybrid trajectories resulting from impulsive dynamics. These computational results

not only validate the analytical theorems but also serve as a blueprint for the implementation of such models in practice.
It is worth noting that while this paper focuses on second-order scalar systems, the techniques employed herein lay the groundwork for more advanced generalizations. Potential future extensions include the treatment of:

- **Multi-dimensional impulsive systems** arising in networked neural populations,
- **Time-delay effects** in impulse-driven control systems,
- **Uncertainty modeling** through fuzzy or neutrosophic logic in high-variability biological environments.

Overall, this study offers a unified analytical-numerical framework that enhances the predictability and robustness of hybrid systems, setting the stage for broader applications in neuroscience, robotics, and control engineering.

CONCLUSIONS

This paper presented a dual-level study on the uniqueness and existence of solutions for second-order differential equations. The first level addressed classical linear homogeneous systems, revisiting fundamental results using energy methods and the Grönwall–Bellman inequality. The second level tackled nonlinear impulsive systems with applications to neural oscillators and hybrid control systems. Using fixed-point theory and Lyapunov-like functions, existence and uniqueness of periodic solutions were rigorously established. MATLAB simulations for both linear and nonlinear models confirmed theoretical predictions, illustrating the practical relevance of the analysis. This hybrid analytical-numerical framework contributes to the modeling of complex systems in neuroscience, bio-prosthetics, and robotics. Future work may explore multi-scale systems, delayed impulses, or fuzzy uncertainty under the same analytical umbrella.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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