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Analytical Results on existence and stability for coupled systems of three 1st order nonlinear differential equations with nonlocal conditions using Banach contraction mapping approach

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نتائج تحليلية حول وجود واستقرار الأنظمة المقترنة لثلاث معادلات تفاضلية غير خطية من الرتبة الأولى مع شروط غير محلية باستخدام أسلوب تحويل بناخ الانكماشي

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Abstract
In this work, a nonlocal-boundary value problem that includes two types of coupled systems of three first-order nonlinear differential equations with nonlocal conditions is introduced. The existence and uniqueness results of solutions for the Cauchy problem are obtained for certain continuous functions on time and the real line space. A local solution is obtained first, followed by a global solution whose proof employs said a contraction principle in the supremum norm. Additionally, the stability of solutions are analyzed. A local stability result is obtained first by employing continuous dependence of local solutions on the space data. In order to illustrate the link between uniform stability and global stability, in the last part of the paper we focus on a stability for global solutions. In particular, our analysis reveals that the global solutions are uniformly globally stable.

Keywords: Nonlocal coupled system, Local and global stability, Nonlinear differential equations, Contraction principle.

الملخص
في هذه الورقة، تم التعامل مع مسألة القيمة الحدية غير المحلية متضمنة نظامين مقترنين يحتويان على ثلاث معادلات تفاضلية غير خطية من الرتبة الأولى مع شروط غير محلية. وقد تم اثبات نتائج وجود ووحدانية الحلول لمسألة كوشي لدوال معينة متصلة و معرّفة على الزمن و مجموعة الأعداد الحقيقة. حيث تم الحصول على الحل محلياً، ثم الحل الشامل، وقد استخدمنا في البرهان مبدأ الانكماش في معيار القيمة العليا. بالإضافة إلى ذلك، تم تحليل استقرار الحلول. وقد تم التوصل إلى نتيجة الاستقرار المحلي أولاً من خلال توظيف الاعتماد المتصل للحلول المحلية على بيانات فضاء بناخ. و نركز في الجزء الأخير من هذه الورقة على استقرار الحلول الشاملة و توضيح العلاقة بين الاستقرار المنتظم والاستقرار الشامل، حيث اظهر التحليل أن الحلول الشاملة مستقرة بشكل منتظم.

الكلمات المفتاحية: نظام اقتران غير محلي، استقرار محلي وشامل، معادلات تفاضلية غير خطية، مبدأ الانكماش.

1. Introduction

In the last decades, much of nonlocal problems have been extensively investigated by many researchers due to its important applications in a variety of engineering and scientific disciplines. For some literature on such problems we refer to [1-4] [6-7] [10-12] [15-17] and the references cited therein. Motivated by the recent works [8- 9], here we fix $T > 0$ and consider the Cauchy nonlocal systems:

$$\frac{dx_1}{dt} = f_1(t, x_2(t)), \quad 0 < t \leq T,$$

$$\frac{dx_2}{dt} = f_1(t, x_3(t)), \quad 0 < t \leq T, \tag{1a}$$

$$\frac{dx_3}{dt}(t) = f_3(t, x_1(t)), \quad 0 < t \leq T,$$

And

$$\frac{dx_1}{dt} = f_1(t, x_3(t)), \quad 0 \leq t \leq T,$$

$$\frac{dx_2}{dt} = f_2(t, x_1(t)), \quad 0 \leq t \leq T, \tag{1b}$$

$$\frac{dx_3}{dt} = f_3(t, x_2(t)), \quad 0 \leq t \leq T,$$

with nonlocal conditions

$$\alpha_1 x_1(\tau_1) + \beta_1 x_1(\gamma_1) = x_1^0, \quad 0 \leq \tau_1 < T, \quad 0 < \gamma_1 \leq T$$

$$\alpha_2 x_2(\tau_2) + \beta_2 x_2(\gamma_2) = x_2^0, \quad 0 \leq \tau_2 < T, \quad 0 < \gamma_2 \leq T, \tag{2}$$

$$\alpha_3 x_3(\tau_3) + \beta_3 x_3(\gamma_3) = x_3^0, \quad 0 \leq \tau_3 < T, \quad 0 < \gamma_3 \leq T$$

where f_i $(i = 1,2,3)$ are given functions and $\alpha_i > 0$, $\beta_i > 0$, $\alpha_i \neq \beta_i$ $\forall (i = 1,2,3)$ are the parameters. The two nonlocal problems (1a)-(2) and (1b)-(2) can be rewritten as following:

$$\frac{dx_i}{dt} = f_i\left(t, x_j(t)\right) \tag{3}$$

$$\alpha_i x_i(\tau_i) + \beta_i x_i(\gamma_i) = x_i^0, \quad \alpha_i \neq \beta_i, i = 1,2,3 \tag{4}$$

where , $i, j = \{1,2,3\}, i \neq j$, $\tau_i \in [0, T), \gamma_i \in (0, T]$, without loss of generality, we will deal with the nonlocal problem (3)-(4).

The aim of this work is to study the existence of at most one local (and, maybe, global) solution for the nonlocal problem (3)-(4). Moreover, the stability of solutions will be established by verifying local and uniform continuous dependence of local and global solutions; respectively, on the space data (i.e., the initial data, parameters, and given system functions). The result of this work show a strong connection between uniform stable property and global stable property of global solutions.

The rest of paper is organized as follows: the next section contains some basic notations and useful definitions needed to prove our main results. This includes very strong regularity assumptions of continuous functions with at most Lipschitz, which, of course, ensures their boundedness through linear growth property. In Section 3, we derive an integral representation for the nonlocal problem (3)-(4). It also covers some basic properties of the integral system. Section 4 is dedicated to present auxiliary lemmas required to establish our main results on the existence and uniqueness of local and global solutions for the nonlocal problem (3)-(4) via the Banach's fixed-point theorem. The stability analysis of local and global solutions are investigated in the last section as well.

2. Preliminary notions

By definition, $C[0, T]$ is the space of all continuous functions $x : [0, T] \to \mathbb{R}$ endowed with the uniform norm defined by $x = sup\{|x(t)|: 0 \leq t \leq T\}$. We define the space $C$of all vectors $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1, x_2, x_3 \in C[0, T]$ equiped with the vector-valued norm given by $\|x\|_C = \sum_{i=1}^{3} \|x_i\|$.

We recall also that, for $\lambda > 0$ be given and $t_0 := \max\{\tau_i, \gamma_i\}, i = 1,2,3$, where $C[t_0, T]$ is the Banach space of all continuous functions x on $[t_0, T]$ endowed with the weighted norm defined by $\|x\|^* = sup\{|x(t)| exp(-\lambda(t - t_0)) : t_0 \leq t \leq T\}$. The vector space C^* of all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_i \in C[t_0, T]$ for $i \in$

{1,2,3} endowed with the vector-valued norm given by $\|x\|^*{}_{C^*} = \sum_{i=1}^{3}\|x_i\|^*$ is also a Banach space. This notations and definitions will be used throughout the forthcoming analysis of this work. The main results will be obtained under the following assumptions:

(A_1) For every $i \in \{1,2,3\}, f_i : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions;

(A_2) For every $0 \le t \le T$, $i \in \{1,2,3\}, f_i : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz continuity

$$\left|f_i(t,x_j) - f_i(t,\breve{x}_j)\right| \le L_i|x_j - \breve{x}_j|, \forall\ x_j, \breve{x}_j \in \mathbb{R}, \text{ where } L_i > 0 \text{ is a constant.} \quad (5)$$

(A_3) $L = \max\{L_i: i = 1,2,3\} < {}^1/_{2T}$

Remark 2.1 (Growth linearity). Observe that the assumption (A_2) above yields the boundedness of f_i on $[0,T] \times C$; indeed, for every $i \in \{1,2,3\}$, there are respectively positive finite constants M_i such that $|f_i(t,x_j)| \le M_i$ for every $0 \le t \le T$ and every $x_j \in C(0,T]$.

Proof. Let us first define $sup_{\,t\in[0,T]} f_i(t,0) = \mu_i < \infty$ for $i \in \{1,2,3\}$ and then apply the inequality (5) for $x_j \in C(0,T]$ with $i \in \{1,2,3\}$ and for $t \in [0,T]$, we get

$$|f_i(t,x_j)| \le |f_i(t,x_j) - f_i(t,0)| + |f_i(t,0)|$$

$$\le L_i|x_j| + \mu_i \le L_i\|x_j\| + \mu_i =: M_i, \forall\, i \in \{1,2,3\} \quad (6)$$

3. Equivalent System Representation

The nonlocal problem (3)-(4) is equivalent to the system of integral equations

$$x_i(t) = a_i\left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s,x_j(s)\right)ds - \beta_i \int_0^{\gamma_i} f_i\left(s,x_j(s)\right)ds\right] + \int_0^t f_i\left(s,x_j(s)\right)ds, \quad (7)$$

where $a_i = (\alpha_i + \beta_i)^{-1}, i \in \{1,2,3\}, 0 < t \le T$.

Theorem 3.1. A vector-valued function x is a local solution of (3)-(4) in C (or, a global solution in C^*) if and only if it can be represented by the integral system (7).

Proof. For the direct implication, we begin by integrating the equations of (3) to get

$$x_i(t) = x_i(0) + \int_0^t f_i\left(s,x_j(s)\right)ds, \quad (8)$$

Then, putting $t = \tau_i$ in equation (8), we get

$$x_i(\tau_i) = x_i(0) + \int_0^{\tau_i} f_i\left(s,x_j(s)\right)ds, \quad (9)$$

Multiplying both sides of (9) by α_i, we get

$$\alpha_i\, x_i(\tau_i) = \alpha_i x_i(0) + \alpha_i \int_0^{\tau_i} f_i\left(s,x_j(s)\right)ds, \quad (10)$$

Similarly, we can get

$$\beta_i\, x_i(\gamma_i) = \beta_i x_i(0) + \beta_i \int_0^{\gamma_i} f_i\left(s,x_j(s)\right)ds, \quad (11)$$

By adding (10) and (11) we obtain that

$$x_i^0 = \alpha_i\, x_i(\tau_i) + \beta_i\, x_i(\gamma_i),$$

which implies that

$$x_i^0 = (\alpha_i + \beta_i)\, x_i(0) + \alpha_i \int_0^{\tau_i} f_i\left(s,x_j(s)\right)ds + \beta_i \int_0^{\gamma_i} f_i\left(s,x_j(s)\right)ds.$$

Therefore,

$$x_i(0) = (\alpha_i + \beta_i)^{-1}\left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right], \quad (12)$$

Substitute (12) in (8), we get

$$x_i(t) = (\alpha_i + \beta_i)^{-1}\left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right] + \int_0^{t} f_i\left(s, x_j(s)\right) ds.$$

Conversely, we only take differentiating the equations of (7) with respect to t that immediately yields the equations of (3). As far as the nonlocal conditions of (4) is concerned, let us first set $t = \tau_i$ in (7) to have

$$x_i(\tau_i) = (\alpha_i + \beta_i)^{-1}\left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right] + \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds.$$

Multiplying the last identity by α_i gives for $i \in \{1,2,3\}$

$$\alpha_i\ x_i(\tau_i) = \alpha_i(\alpha_i + \beta_i)^{-1}\left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right] + \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds. \quad (13)$$

Similarly, we have

$$\beta_i\ x_i(\gamma_i) = \beta_i(\alpha_i + \beta_i)^{-1}\left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right] + \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds. \quad (14)$$

Finally, we combine (13) and (14) to deduce (4), and thus the proof is finished.

In light of the fact that the functions $f_i, i \in \{1,2,3\}$ make the integral system (7) exhibits non-decreasing monotonicity in time $t \in [0, T]$, we state the following corollary.

Corollary 3.2. Suppose for $i \in \{1,2,3\}$, $f_i: [0, T] \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfy the assumption(A_1). Then the solutions (7) of the nonlocal problem (3)-(4) are monotone non-decreasing in time.

Proof. Let the solutions $x_i$of (3)-(4) be given by (7), then for $t_1 \leq t_2$ we have

$$x_i(t_1) = a_i\left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right] + \int_0^{t_1} f_i\left(s, x_j(s)\right) ds$$

$$\leq a_i\left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right] + \int_0^{t_2} f_i\left(s, x_j(s)\right) ds$$

$$= x_i(t_2).$$

Under the same assumption as in Corollary3.2, we also have the non-negativity of solutions.

Corollary3.3. Suppose for $i \in \{1,2,3\}$, $f_i: [0, T] \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$satisfy the assumption (A_1). Then the solutions (7) of the nonlocal problem (3)-(4) are nonnegative for every time.

Proof. Let the solutions x_i of (3)-(4) be given by (7) with $x_i^0 \geq 0$. Then, for $0 < t \leq T$, we have

$\int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds \leq \int_0^{t} f_i\left(s, x_j(s)\right) ds$ for some $\tau_i \leq t$, $i \in \{1,2,3\}$,

and hence

$$\alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds \leq \alpha_i \int_0^{t} f_i\left(s, x_j(s)\right) ds,$$

multiplying by $a_i = (\alpha_i + \beta_i)^{-1}$, we get

$a_i\alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds \leq a_i\alpha_i \int_0^{t} f_i\left(s, x_j(s)\right) ds,$

which implies that

$a_i\alpha_i \int_0^{t} f_i\left(s, x_j(s)\right) ds - a_i\alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds \geq 0 .$ (15)

Similarly, we have

$a_i\beta_i \int_0^{t} f_i\left(s, x_j(s)\right) ds - a_i\beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds \geq 0$ (16)

We combine (15) and (16) to obtain that

$$a_i(\alpha_i + \beta_i) \int_0^t f_i\left(s, x_j(s)\right) ds - a_i\alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - a_i\beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds \geq 0 ,$$

which leads to

$$\int_0^t f_i\left(s, x_j(s)\right) ds - a_i\alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - a_i\beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds \geq 0$$

Since $x_i^0 \geq 0$ and $a_i > 0$, we have

$$a_i \left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right] + \int_0^t f_i\left(s, x_j(s)\right) ds \geq 0$$

which implies that $x_i(t) \geq 0$.

4. Main Results: Existence and Uniqueness of Solutions

We shall seek the existence of at most one solution of nonlocal problem (3)-(4). To do that, let us first define the vector-valued super position $\mathcal{H}$ on C, given by
$\mathcal{H}(x)(t) := (\mathcal{H}_i x_j(t): i = 1,2,3)$, where $x_j(t) \in C, j \in \{1,2,3\}, i \neq j$ and
$\mathcal{H}_i x_j(t) := a_i \left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right] + \int_0^t f_i\left(s, x_j(s)\right) ds.$ (17)

Lemma 4.1. Under the assumption(A_1), the operator $\mathcal{H}$ maps C into itself.
Proof. Note first that $\mathcal{H}$ is continuous map by the time-continuity of f_i , $i \in \{1,2,3\}$ as compositions of continuous scalar functions of $t \in [0, T]$, which; indeed, is uniformly bounded by some positive constant $M := max\{M_i : i = 1,2,3\} < \infty$, where
$M_i = sup\{|f_i(t, x_j(t))|: 0 \leq t \leq T\}$. Moreover, let $x_j(t) \in C(0, T]$ and without loss of generality we assume $t \in (0, T]$ Then, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > t$ such that

$$\left|\mathcal{H}_i x_j(t) - \mathcal{H}_i x_j(0)\right| \leq \int_0^t \left|f_i\left(s, x_j(s)\right)\right| ds \leq \int_0^t M_i \, ds = M_i t < \frac{\varepsilon}{3}$$

Thus, by taking $\delta := \frac{\varepsilon}{3M}$ we infer $|\mathcal{H}(x)(t) - \mathcal{H}(x)(0)| < \varepsilon$ and hence $\mathcal{H}$ takes value in C.
We shall next state a lemma, which carries another fact to our context.

Lemma 4.2. Under the assumption(A_1), the operator $\mathcal{H}$ is uniformly bounded on C.
Proof. We employ the linear growth (6) for any $x_j(t) \in C(0, T]$ and any $t \in [0, T]$, to get

$$\begin{aligned}\left|\mathcal{H}_i x_j(t)\right| &= \left| a_i \left[x_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds\right] + \int_0^t f_i\left(s, x_j(s)\right) ds \right|, \\ &\leq a_i\left|x_i^0\right| + a_i\alpha_i \int_0^{\tau_i} \left|f_i\left(s, x_j(s)\right)\right| ds + a_i\beta_i \int_0^{\gamma_i} \left|f_i\left(s, x_j(s)\right)\right| ds + \int_0^t \left|f_i\left(s, x_j(s)\right)\right| ds, \\ &\leq a_i\left|x_i^0\right| + a_i\alpha_i \int_0^{T} \left|f_i\left(s, x_j(s)\right)\right| ds + a_i\beta_i \int_0^{T} \left|f_i\left(s, x_j(s)\right)\right| ds + \int_0^T \left|f_i\left(s, x_j(s)\right)\right| ds,\end{aligned}$$

$$\leq a_i|x_i^0| + [\,a_i(\alpha_i+\beta_i)+1]\int_0^T \left|f_i\left(s,x_j(s)\right)\right| ds,$$
$$\leq a_i|x_i^0| + 2\int_0^T (L_i\|x_j\| + \mu_i)ds,$$
$$\leq a_i|x_i^0| + 2T(L_i\|x_j\| + \mu_i),$$
$$\leq a_i|x_i^0| + 2TM_i\,,$$
which leads to $\|\mathcal{H}_i x_j(t)\| \leq a_i|x_i^0| + 2TM_i =: K_i\,, i = 1,2,3$. It follows that
$$\|\mathcal{H}(x)\|_C \leq \sum_1^3 K_i =: K$$
Which proves the boundedness of $\mathcal{H}(C) \subset C$.
As a direct consequence of Lemma 4.2, we obtain the following corollary.

Corollary 4.3. Under the assumptions$(A_1)-(A_3)$, we define the closed subset $C_r = \{x \in C : \|x\|_C \leq r \; for\; some\; r > 0\}$, so that Lemmas 4.1, 4.2 are applicable.
Proof. From the proof argument of Lemma 4.2 above it must have
$$r > \frac{\sum_{i=1}^3(\,a_i|x_i^0| + 2TM_i)}{1-2TL}$$
In particular, one may take
$$r = \frac{\sum_{i=1}^3(\,a_i|x_i^0| + 2TM_i)}{1-2TL} = \frac{\sum_1^3 K_i}{1-2TL} = \frac{K}{1-2TL}.$$

Remark 4.4. By the very similar arguments to the above proofs we can conclude the same results as in Lemmas 4.1, 4.2 hold also (under the relevant global assumptions) on the space C^*, whose definition was indeed modeled on the equivalent norm. Anyway, we will come back later on the space $C^*$and show that, in fact, it possesses a single global solution for (3)-(4), which exhibits uniform stability and global stability simultaneously.

4.1. Local solution in the C-space

The first main result for the existence of only one local solution in the space C follows as:

Theorem 4.5. Suppose the assumptions$(A_1)-(A_3)$ hold locally in C . Then, the nonlocal problem (3)-(4) admits at most one local solution given by (7) in C.
Proof. Let $x \in C$ and let $\mathcal{H}(x)(t)$ be given by (17). For $\breve{x} \in C$ and any $t \in [0,T]$, we have $\mathcal{H}(\breve{x})(t) = (\mathcal{H}_i\breve{x}_j(t): i = 1,2,3)$, (18)
where $(\breve{x}_j(t) \in C: j = 1,2,3)$ and
$$\mathcal{H}_i\breve{x}_j(t) = a_i\left[x_i^0 - \alpha_i\int_0^{\tau_i} f_i\left(s,\breve{x}_j(s)\right)ds - \beta_i\int_0^{\gamma_i} f_i\left(s,\breve{x}_j(s)\right)ds\right] + \int_0^t f_i\left(s,\breve{x}_j(s)\right)ds. \quad (19)$$
Then, from (17) and (19), we deduce that
$$\left|\mathcal{H}_i x_j(t) - \mathcal{H}_i\breve{x}_j(t)\right|$$
$$\leq a_i\alpha_i\int_0^{\tau_i}\left|f_i\left(s,x_j(s)\right) - f_i\left(s,\breve{x}_j(s)\right)\right|ds + a_i\beta_i\int_0^{\gamma_i}\left|f_i\left(s,x_j(s)\right) - f_i\left(s,\breve{x}_j(s)\right)\right|ds$$
$$+\int_0^t\left|f_i\left(s,x_j(s)\right) - f_i\left(s,\breve{x}_j(s)\right)\right|ds$$
Using (A_2) we get
$$\left|\mathcal{H}_i x_j(t) - \mathcal{H}_i\breve{x}_j(t)\right| \leq a_i\alpha_i\int_0^{\tau_i} L_i|x_j - \breve{x}_j|\,ds + a_i\beta_i\int_0^{\gamma_i} L_i|x_j - \breve{x}_j|\,ds + \int_0^t L_i|x_j - \breve{x}_j|\,ds$$
Taking the supremum and integrating on $[0,T]$ leads to
$$\left\|\mathcal{H}_i x_j(t) - \mathcal{H}_i\breve{x}_j(t)\right\| \leq 2\,T\,L_i\|x_j - \breve{x}_j\|$$
It follows from locally(A_3), in C that one can estimate
$$\|\mathcal{H}(x) - \mathcal{H}(\breve{x})\| \leq 2\,TL\|x - \breve{x}\|$$
Which proves that the local operator $\mathcal{H}: C \to C$ is a contraction. Applying the local Banach's contraction mapping principle [13-14] for the existence of a unique fixed local solution to (7) we deduce from theorem (3.1), that the nonlocal problem (3)-(4) has a unique local solution in C.

4.2. Global solution in the C^*-space

The second main result for the existence of unique global solution in the space $C^*$is the following:

Theorem 4.6. Suppose the assumptions$(A_1)-(A_3)$ hold globally in C^*. Then, the nonlocal problem (3)-(4) has a unique global solution given by (3.1) in C^*.

Proof. Let $x, \breve{x} \in C$ and let $\mathcal{H}(x), \mathcal{H}(\breve{x})$ be given respectively by (17) and (18). Then, for $x_j, \breve{x}_j \in C[t_0, T]$ and for any $t \in [t_0, T]$, we deduce for every $i \in \{1,2,3\}$

$$\left|\mathcal{H}_i x_j(t) - \mathcal{H}_i \breve{x}_j(t)\right|$$
$$\leq a_i\alpha_i \int_0^{\tau_i} \left|f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right)\right| e^{-\lambda(s-t_0)} e^{\lambda(s-t_0)} ds$$
$$+ a_i\beta_i \int_0^{\gamma_i} \left|f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right)\right| e^{-\lambda(s-t_0)} e^{\lambda(s-t_0)} ds$$
$$+ \int_0^{t} \left|f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right)\right| e^{-\lambda(s-t_0)} e^{\lambda(s-t_0)} ds$$

and, by utilizing (A_2) on$[t_0, T]$, we obtain

$$\left|\mathcal{H}_i x_j(t) - \mathcal{H}_i \breve{x}_j(t)\right|$$
$$\leq a_i\alpha_i \int_0^{\tau_i} L_i\left|x_j - \breve{x}_j\right| e^{-\lambda(s-t_0)} e^{\lambda(s-t_0)}\, ds + a_i\beta_i \int_0^{\gamma_i} L_i\left|x_j - \breve{x}_j\right| e^{-\lambda(s-t_0)} e^{\lambda(s-t_0)}\, ds$$
$$+ \int_0^{t} L_i\left|x_j - \breve{x}_j\right| e^{-\lambda(s-t_0)} e^{\lambda(s-t_0)} ds$$

Multiplying by $e^{-\lambda(t-t_0)}$and taking the supremum when $t \in [t_0, T]$ leads to

$$\left\|\mathcal{H}_i x_j(t) - \mathcal{H}_i \breve{x}_j(t)\right\|^*$$
$$\leq a_i\alpha_i\, L_i\left\|x_j - \breve{x}_j\right\|^* \int_0^{\tau_i} e^{\lambda(s-t)} ds + a_i\beta_i L_i\left\|x_j - \breve{x}_j\right\|^* \int_0^{\gamma_i} e^{\lambda(s-t)}\, ds$$
$$+ L_i\left\|x_j - \breve{x}_j\right\|^* \int_0^{t} e^{\lambda(s-t)}\, ds$$

After integrating, we have for $\lambda > 0$

$$\left\|\mathcal{H}_i x_j(t) - \mathcal{H}_i \breve{x}_j(t)\right\|^*$$
$$\leq \frac{1}{\lambda} a_i\alpha_i\, L_i\left\|x_j - \breve{x}_j\right\|^* \left(e^{-\lambda(t-\tau_i)} - e^{-\lambda t}\right) + \frac{1}{\lambda}\, a_i\beta_i L_i\left\|x_j - \breve{x}_j\right\|^* \left(e^{-\lambda(t-\gamma_i)} - e^{-\lambda t}\right)$$
$$+ \frac{1}{\lambda}\, L_i\left\|x_j - \breve{x}_j\right\|^* \left(1 - e^{-\lambda t}\right)$$

Then we can write for any $t \in [t_0, T]$,

$$\left\|\mathcal{H}_i x_j(t) - \mathcal{H}_i \breve{x}_j(t)\right\|^*$$
$$\leq \frac{1}{\lambda} a_i\alpha_i\, L_i\left\|x_j - \breve{x}_j\right\|^* \left(e^{-\lambda(t-t_0)} - e^{-\lambda t}\right) + \frac{1}{\lambda}\, a_i\beta_i L_i\left\|x_j - \breve{x}_j\right\|^* \left(e^{-\lambda(t-t_0)} - e^{-\lambda t}\right)$$
$$+ \frac{1}{\lambda}\, L_i\left\|x_j - \breve{x}_j\right\|^* \left(1 - e^{-\lambda t}\right)$$

which implies that

$$\left\|\mathcal{H}_i x_j(t) - \mathcal{H}_i \breve{x}_j(t)\right\|^* \leq \frac{1}{\lambda} L_i\left\|x_j - \breve{x}_j\right\|^* \left(e^{-\lambda(t-t_0)} - e^{-\lambda t}\right) + \frac{1}{\lambda}\, L_i\left\|x_j - \breve{x}_j\right\|^* \left(1 - e^{-\lambda t}\right)$$
$$\leq \tfrac{1}{\lambda} L_i\left\|x_j - \breve{x}_j\right\|^* + \tfrac{1}{\lambda}\, L_i\left\|x_j - \breve{x}_j\right\|^* \left(1 - e^{-\lambda t}\right)$$
$$\leq \frac{2}{\lambda} L_i\left\|x_j - \breve{x}_j\right\|^* - \frac{1}{\lambda}\, L_i\left\|x_j - \breve{x}_j\right\|^* e^{-\lambda t}$$
$$\left\|\mathcal{H}_i x_j(t) - \mathcal{H}_i \breve{x}_j(t)\right\|^* \leq \frac{2}{\lambda} L_i\left\|x_j - \breve{x}_j\right\|^*$$

Consequently, by using (A_3)one can estimate

$$\|\mathcal{H}(x) - \mathcal{H}(\breve{x})\|_{C^*} \leq \frac{2L}{\lambda} \|x - \breve{x}\|_{C^*}$$

for sufficiently large $\lambda > 2L$ which yields the global operator $\mathcal{H}: C^* \to C^*$ is a contraction (due to the global fixed point theorem [13, 14]), hence, according to Theorem 3.1, the nonlocal problem (3)-(4) has at most one global solution given by (7) in C^*.

5. Stability Analysis of Solutions

We will study the stability of local and global solutions for the nonlocal problem (3)-(4).

5.1. Stability of local solutions

We begin investigating a stability of the local solution $x \in C$ for the nonlocal problem (3)-(4) by showing its local continuously dependence on the space data (i.e., the initial data, parameters, and given system functions).

5.1.1 Local continuous dependence on the initial data

For $i \in \{1,2,3\}$ we consider the nonlocal problem

$$\frac{d\breve{x}_i}{dt} = f_i\left(t, \breve{x}_j(t)\right) \quad (20)$$

$$\alpha_i \breve{x}_i(\tau_i) + \beta_i \breve{x}_i(\gamma_i) = \breve{x}_i^0, \quad \alpha_i \neq \beta_i, i = 1,2,3 \quad (21)$$

where $\breve{x}_i^0$ are the initial data. Let us first recall the following definition.

Definition 5.1. We say that the local solution $x \in C$ of the nonlocal problem (3)-(4) is locally continuous dependence on $\breve{x}^0 = (\breve{x}_i^0 : i = 1,2,3)$ if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$, dependent on times, such that $|x^0 - \breve{x}^0| \leq \delta$ implies $\|x - \breve{x}\|_C \leq \varepsilon$, where $\breve{x} \in C$ is the local solution of (20)-(21).

Based on this criterion, we establish the following statement.

Theorem 5.2. Suppose as in Theorem 4.5 that $(A_1) - (A_3)$ hold locally in C. Then, the unique local solution of (3)-(4) in C depends local continuously on the initial data.

Proof. Let, due to Theorems 3.1 and 4.5, the unique local solution $x \in C$ of (3)-(4) be given by (7) and, further, let

$$\breve{x}_i(t) = a_i \left[\breve{x}_i^0 - \alpha_i \int_0^{\tau_i} f_i\left(s, \breve{x}_j(s)\right) ds - \beta_i \int_0^{\gamma_i} f_i\left(s, \breve{x}_j(s)\right) ds\right] + \int_0^t f_i\left(s, \breve{x}_j(s)\right) ds,$$

be the unique local solution of (20)-(21) in C. Then, we have

$$|x_i(t) - \breve{x}_i(t)| \leq a_i |x_i^0 - \breve{x}_i^0| + a_i \alpha_i \int_0^{\tau_i} \left|f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right)\right| ds + a_i \beta_i \int_0^{\gamma_i} \left|f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right)\right| ds + \int_0^t \left|f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right)\right| ds$$

$$|x_i(t) - \breve{x}_i(t)| \leq a_i |x_i^0 - \breve{x}_i^0| + a_i \alpha_i \int_0^{\tau_i} L_i |x_j - \breve{x}_j| ds + a_i \beta_i \int_0^{\gamma_i} L_i |x_j - \breve{x}_j| ds + \int_0^t L_i |x_j - \breve{x}_j| ds$$

Taking the supremum for $t \in [0, T]$, and integrating over time we deduce that

$$\|x_i - \breve{x}_i\| \leq a_i \frac{\delta}{3} + 2L_i T \|x_j - \breve{x}_j\|$$

Then,

$$\|x - \breve{x}\| \leq \frac{\delta}{3} \sum_{i=1}^{3} a_i + 2LT \|x - \breve{x}\|$$

Therefore, we obtain

$$\|x - \breve{x}\| \leq \frac{\sum_{i=1}^{3} a_i}{1 - 2LT} \frac{\delta}{3} \leq \varepsilon$$

Whenever $L < \frac{1}{2T}$.

5.1.2 Local continuous dependence on the parameters

For $i \in \{1,2,3\}$, we consider the nonlocal problem

$$\frac{d\breve{x}_i}{dt} = f_i\left(t, \breve{x}_j(t)\right) \quad (22)$$

$$\breve{\alpha}_i \breve{x}_i(\tau_i) + \breve{\beta}_i \breve{x}_i(\gamma_i) = x_i^0, \quad \breve{\alpha}_i \neq \breve{\beta}_i, i, j = 1,2,3;\ i \neq j \quad (23)$$

Definition 5.3. We say that the local solution $x \in C$ of the nonlocal problem (3)-(4) is locally continuous dependence on α, β if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$, dependent on times, such that $|\alpha - \breve{\alpha}| \le \delta_1$, $|\beta - \breve{\beta}| \le \delta_2$ implies $\|x - \breve{x}\|_C \le \varepsilon$ where $\delta = \max\{\delta_1, \delta_2\}$ and $\breve{x} \in C$ is the local solution of (22)-(23).

Theorem5.4. Suppose as in Theorem 4.5 that $(A_1) - (A_3)$ hold locally in C. Then, the unique local solution of nonlocal problem (3)-(4) in C depends local continuously on the parameters.

Proof. Let the unique local solution $x \in C$ of (3)-(4) given by (7) and let $\breve{x} \in C$ be the unique local solution of (22)-(23) given by

$$\breve{x}_i(t) = \breve{a}_i\left[\breve{x}_i^0 - \breve{\alpha}_i\int_0^{\tau_i} f_i\left(s, \breve{x}_j(s)\right) ds - \breve{\beta}_i\int_0^{\gamma_i} f_i\left(s, \breve{x}_j(s)\right) ds\right] + \int_0^t f_i\left(s, \breve{x}_j(s)\right) ds,$$

where $\breve{a}_i = (\breve{\alpha}_i + \breve{\beta}_i)^{-1}$, $i = 1,2,3$. Then we have

$$\begin{aligned}
|x_i(t) - \breve{x}_i(t)| = \Bigg| & a_i x_i^0 - a_i\alpha_i\int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - a_i\beta_i\int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds \\
& + \int_0^t f_i\left(s, x_j(s)\right) ds - \breve{a}_i\breve{x}_i^0 + \breve{a}_i\breve{\alpha}_i\int_0^{\tau_i} f_i\left(s, \breve{x}_j(s)\right) ds + \breve{a}_i\breve{\beta}_i\int_0^{\gamma_i} f_i\left(s, \breve{x}_j(s)\right) ds \\
& - \int_0^t f_i\left(s, \breve{x}_j(s)\right) ds \Bigg| \\
= \Bigg| & a_i x_i^0 - a_i\alpha_i\int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - a_i\beta_i\int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds \\
& + a_i\alpha_i\int_0^{\tau_i} f_i\left(s, \breve{x}_j(s)\right) ds - a_i\alpha_i\int_0^{\tau_i} f_i\left(s, \breve{x}_j(s)\right) ds \\
& + \int_0^t f_i\left(s, x_j(s)\right) ds - \breve{a}_i x_i^0 \\
& + \breve{a}_i\breve{\alpha}_i\int_0^{\tau_i} f_i\left(s, \breve{x}_j(s)\right) ds \\
& + \breve{a}_i\breve{\beta}_i\int_0^{\gamma_i} f_i\left(s, \breve{x}_j(s)\right) ds - \breve{a}_i\breve{\beta}_i\int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds + \breve{a}_i\breve{\beta}_i\int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds \\
& - \int_0^t f_i\left(s, \breve{x}_j(s)\right) ds \Bigg| \\
\le\ & |a_i - \breve{a}_i| x_i^0 \\
& + a_i\alpha_i\int_0^{\tau_i}\left|f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right)\right| ds + \breve{a}_i\breve{\beta}_i\int_0^{\gamma_i}\left|f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right)\right| ds \\
& + \int_0^t\left|f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right)\right| ds + |a_i\beta_i - \breve{a}_i\breve{\beta}_i|\int_0^{\gamma_i}\left|f_i\left(s, x_j(s)\right)\right| ds \\
& + |a_i\alpha_i - \breve{a}_i\breve{\alpha}_i|\int_0^{\tau_i}\left|f_i\left(s, \breve{x}_j(s)\right)\right| ds \\
\le\ & \frac{|\alpha_i - \breve{\alpha}_i| + |\beta_i - \breve{\beta}_i|}{(\alpha_i + \beta_i)(\breve{\alpha}_i + \breve{\beta}_i)}\left|x_i^0\right| \\
& + a_i\alpha_i L_i\int_0^{\tau_i}\left|x_j(s) - \breve{x}_j(s)\right| ds + \breve{a}_i\breve{\beta}_i L_i\int_0^{\gamma_i}\left|x_j(s) - \breve{x}_j(s)\right| ds + L_i\int_0^t\left|x_j(s) - \breve{x}_j(s)\right| ds \\
& + \frac{|\breve{\alpha}_i\beta_i - \alpha_i\breve{\beta}_i|}{(\alpha_i + \beta_i)(\breve{\alpha}_i + \breve{\beta}_i)}\int_0^{\gamma_i}\left|f_i\left(s, x_j(s)\right)\right| ds + \frac{|\alpha_i\breve{\beta}_i - \breve{\alpha}_i\beta_i|}{(\alpha_i + \beta_i)(\breve{\alpha}_i + \breve{\beta}_i)}\int_0^{\tau_i}\left|f_i\left(s, \breve{x}_j(s)\right)\right| ds \\
\le\ & \frac{|\alpha_i - \breve{\alpha}_i| + |\beta_i - \breve{\beta}_i|}{(\alpha_i + \beta_i)(\breve{\alpha}_i + \breve{\beta}_i)}\left|x_i^0\right| \\
& + a_i\alpha_i L_i\int_0^{T}\left|x_j(s) - \breve{x}_j(s)\right| ds + \breve{a}_i\breve{\beta}_i L_i\int_0^{T}\left|x_j(s) - \breve{x}_j(s)\right| ds + L_i\int_0^T\left|x_j(s) - \breve{x}_j(s)\right| ds \\
& + \frac{|\breve{\alpha}_i\beta_i - \alpha_i\breve{\beta}_i|}{(\alpha_i + \beta_i)(\breve{\alpha}_i + \breve{\beta}_i)}\int_0^{T} M_i\, ds + \frac{|\alpha_i\breve{\beta}_i - \breve{\alpha}_i\beta_i|}{(\alpha_i + \beta_i)(\breve{\alpha}_i + \breve{\beta}_i)}\int_0^{T} M_i\, ds
\end{aligned}$$

$$|x_i(t) - \breve{x}_i(t)| \le \frac{|\alpha_i - \breve{\alpha}_i| + |\beta_i - \breve{\beta}_i|}{(\alpha_i+\beta_i)(\breve{\alpha}_i+\breve{\beta}_i)}|x_i^0| + \frac{\alpha_i}{(\alpha_i+\beta_i)} L_i T \|x_j - \breve{x}_j\| + \frac{\breve{\beta}_i}{(\breve{\alpha}_i+\breve{\beta}_i)} L_i T \|x_j - \breve{x}_j\|$$
$$+ L_i T \|x_j - \breve{x}_j\| + \frac{2|\breve{\alpha}_i\beta_i - \alpha_i\breve{\beta}_i + \breve{\alpha}_i\breve{\beta}_i - \breve{\alpha}_i\breve{\beta}_i|}{(\alpha_i+\beta_i)(\breve{\alpha}_i+\breve{\beta}_i)} M_i T$$
$$\le \frac{|\alpha_i - \breve{\alpha}_i| + |\beta_i - \breve{\beta}_i|}{(\alpha_i+\beta_i)(\breve{\alpha}_i+\breve{\beta}_i)}|x_i^0| + \left(\frac{\alpha_i}{(\alpha_i+\beta_i)} + \frac{\breve{\beta}_i}{(\breve{\alpha}_i+\breve{\beta}_i)} + 1\right) L_i T \|x_j - \breve{x}_j\| + \frac{2|\breve{\alpha}_i(\beta_i - \breve{\beta}_i) - \breve{\beta}_i(\alpha_i - \breve{\alpha}_i)|}{(\alpha_i+\beta_i)(\breve{\alpha}_i+\breve{\beta}_i)} M_i T$$
$$\le \frac{|\alpha_i - \breve{\alpha}_i| + |\beta_i - \breve{\beta}_i|}{(\alpha_i+\beta_i)(\breve{\alpha}_i+\breve{\beta}_i)}|x_i^0| + \left(\frac{\alpha_i}{(\alpha_i+\beta_i)} + \frac{\breve{\beta}_i}{(\breve{\alpha}_i+\breve{\beta}_i)} + 1\right) L_i T \|x_j - \breve{x}_j\| + \frac{2\breve{\alpha}_i|\beta_i - \breve{\beta}_i| + 2\breve{\beta}_i|\alpha_i - \breve{\alpha}_i|}{(\alpha_i+\beta_i)(\breve{\alpha}_i+\breve{\beta}_i)} M_i T$$

By taking the supremum on $[0, T]$, we get

$$\|x_i - \breve{x}_i\| \le \frac{(\delta_1+\delta_2)}{3(\alpha_i+\beta_i)(\breve{\alpha}_i+\breve{\beta}_i)}|x_i^0| + \left(\frac{\alpha_i}{(\alpha_i+\beta_i)} + \frac{\breve{\beta}_i}{(\breve{\alpha}_i+\breve{\beta}_i)} + 1\right) L_i T \|x_j - \breve{x}_j\|$$
$$+ \frac{2(\delta_1\breve{\alpha}_i + \delta_2\breve{\beta}_i)}{3(\alpha_i+\beta_i)(\breve{\alpha}_i+\breve{\beta}_i)} M_i T$$

Set $b_i = \frac{\alpha_i}{(\alpha_i+\beta_i)} + \frac{\breve{\beta}_i}{(\breve{\alpha}_i+\breve{\beta}_i)} + 1$, $b = \sum_{i=1}^{3} b_i$, $n_1 = \sum_{i=1}^{3} \breve{\alpha}_i$, $n_2 = \sum_{i=1}^{3} \breve{\beta}_i$ and

$$n_3 = \sum_{i=1}^{3} \frac{1}{(\alpha_i+\beta_i)(\breve{\alpha}_i+\breve{\beta}_i)} , \ M := max\{M_i : i = 1,2,3\}.$$

Hence, by combining the last inequalities for $i = 1,2,3$ together with using (A_3), one can estimate

$$\|x - \breve{x}\| \le \frac{(\delta_1+\delta_2)}{3}|x^0| n_3 + b\, LT \|x - \breve{x}\| + \frac{2\delta_1}{3} MT n_1 n_3 + \frac{2\delta_2}{3} MT n_2 n_3$$
$$(1 - b\, LT)\|x - \breve{x}\| \le \frac{\delta_1 n_3}{3}(|x^0| + 2MTn_1) + \frac{\delta_2 n_3}{3}(|x^0| + 2MTn_2)$$
$$(1 - b\, LT)\|x - \breve{x}\| \le \frac{\delta n_3}{3}(|x^0| + 2MTn_1) + \frac{\delta n_3}{3}(|x^0| + 2MTn_2)$$

Which implies if $L < \frac{1}{bT}$ then

$$\|x - \breve{x}\| \le \frac{2\delta n_3(|x^0| + MT(n_1+n_2))}{3(1 - b\, LT)} < \varepsilon$$

5.1.3 Local continuous dependence on system functions

For $i \in \{1,2,3\}$ we consider the following nonlocal problem

$$\frac{d\breve{x}_i}{dt} = \breve{f}_i\left(t, \breve{x}_j(t)\right) \quad (24)$$
$$\alpha_i \breve{x}_i(\tau_i) + \beta_i \breve{x}_i(\gamma_i) = x_i^0, \quad \alpha_i \ne \beta_i, i = 1,2,3 \quad (25)$$

where $\breve{f}_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given system functions. We then recall the definition.

Definition 5.5. We say that the local solution $x \in C$ of the nonlocal problem (3)-(4) is locally continuous dependence on $f = (f_i : i = 1,2,3)$. If for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$, dependent on times, such that $\|f - \breve{f}\| \le \delta$ implies $\|x - \breve{x}\|_C \le \varepsilon$, where $\breve{x} \in C$ is the local solution of (24)-(25).

Theorem 5.6. Suppose as in Theorem 4.5 that $(A_1) - (A_3)$ hold locally in C. Then, the unique local solution of nonlocal problem (3)-(4) in C depends local continuously on the system functions.

Proof. Let the unique local solution $x \in C$ of (3)-(4) given by (7) and let $\breve{x} \in C$ be the unique local solution of (24)-(25) given by

$$\breve{x}_j(t) = a_i\left[x_i^0 - \alpha_i \int_0^{\tau_i} \breve{f}_i\left(s, \breve{x}_j(s)\right) ds - \beta_i \int_0^{\gamma_i} \breve{f}_i\left(s, \breve{x}_j(s)\right) ds\right] + \int_0^t \breve{f}_i\left(s, \breve{x}_j(s)\right) ds.$$

Then, we have

$$|x_i(t) - \breve{x}_i(t)| = \left| a_i x_i^0 - a_i\alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - a_i\beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds \right.$$
$$+ \int_0^t f_i\left(s, x_j(s)\right) ds - a_i x_i^0 + a_i\alpha_i \int_0^{\tau_i} \breve{f}_i\left(s, \breve{x}_j(s)\right) ds + a_i\beta_i \int_0^{\gamma_i} \breve{f}_i\left(s, \breve{x}_j(s)\right) ds$$
$$\left. - \int_0^t \breve{f}_i\left(s, \breve{x}_j(s)\right) ds \right|$$

$$
\begin{aligned}
|x_i(t) - \breve{x}_i(t)| = \Big| &- a_i\alpha_i \int_0^{\tau_i} f_i\left(s, x_j(s)\right) ds - a_i\beta_i \int_0^{\gamma_i} f_i\left(s, x_j(s)\right) ds \\
&+ a_i\alpha_i \int_0^{\tau_i} f_i\left(s, \breve{x}_j(s)\right) ds + a_i\beta_i \int_0^{\gamma_i} f_i\left(s, \breve{x}_j(s)\right) ds \\
&- a_i\alpha_i \int_0^{\tau_i} f_i\left(s, \breve{x}_j(s)\right) ds - a_i\beta_i \int_0^{\gamma_i} f_i\left(s, \breve{x}_j(s)\right) ds \\
&+ \int_0^{t} f_i\left(s, x_j(s)\right) ds - \int_0^{t} f_i\left(s, \breve{x}_j(s)\right) ds + \int_0^{t} f_i\left(s, \breve{x}_j(s)\right) ds \\
&+ a_i\alpha_i \int_0^{\tau_i} \breve{f}_i\left(s, \breve{x}_j(s)\right) ds + a_i\beta_i \int_0^{\gamma_i} \breve{f}_i\left(s, \breve{x}_j(s)\right) ds - \int_0^{t} \breve{f}_i\left(s, \breve{x}_j(s)\right) ds \Big| \\
\le\; & a_i\alpha_i \int_0^{\tau_i} \left| f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right) \right| ds \\
&+ a_i\alpha_i \int_0^{\tau_i} \left| \breve{f}_i\left(s, \breve{x}_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right) \right| ds \\
&+ a_i\beta_i \int_0^{\gamma_i} \left| f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right) \right| ds \\
&+ a_i\beta_i \int_0^{\gamma_i} \left| \breve{f}_i\left(s, \breve{x}_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right) \right| ds + \int_0^{t} \left| f_i\left(s, x_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right) \right| ds \\
&+ \int_0^{t} \left| \breve{f}_i\left(s, \breve{x}_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right) \right| ds \\
\le\; & a_i\alpha_i L_i \int_0^{\tau_i} |x_j(s) - \breve{x}_j(s)| ds \\
&+ a_i\alpha_i \int_0^{\tau_i} \left| \breve{f}_i\left(s, \breve{x}_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right) \right| ds + a_i\beta_i L_i \int_0^{\gamma_i} |x_j(s) - \breve{x}_j(s)| ds \\
&+ a_i\beta_i \int_0^{\gamma_i} \left| \breve{f}_i\left(s, \breve{x}_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right) \right| ds + L_i \int_0^{t} |x_j(s) - \breve{x}_j(s)| ds \\
&+ \int_0^{t} \left| \breve{f}_i\left(s, \breve{x}_j(s)\right) - f_i\left(s, \breve{x}_j(s)\right) \right| ds
\end{aligned}
$$

By taking the supremum and then integrating over $[0, T]$we get

$$
\|x_i - \breve{x}_i\| \le (a_i\alpha_i + a_i\beta_i + 1)L_i T \|x_j - \breve{x}_j\| + (a_i\alpha_i + a_i\beta_i + 1)T \|f_i - \breve{f}_i\|
$$

Which implies

$$
\|x_i - \breve{x}_i\| \le 2L_i T \|x_j - \breve{x}_j\| + 2T \|f_i - \breve{f}_i\|
$$

Thus,

$$
\|x - \breve{x}\| \le 2LT \|x - \breve{x}\| + 2T \|f - \breve{f}\|
$$

If $L < \frac{1}{2T}$, then we get

$$
\|x - \breve{x}\| \le \frac{2T\delta}{1 - 2LT} < \varepsilon
$$

The next result is a direct consequence of Theorems 5.2, 5.4, and 5.6 stated above.

Corollary 5.7. Assume as in Theorem 4.5 that $(A_1) - (A_3)$ are satisfied locally in C. Then, the unique local solution of (3)-(4) is locally Lyapunov stable (LS) in C.

5.2 Stability of global solutions

We next examine the uniform stability of $x \in C^*$ for the nonlocal problem (3)-(4) by verifying its uniform continuous dependence on the space data. To do this, we recall Definition5.1 as follows:

Definition 5.8. We say that the global solution $x \in C^*$ of the nonlocal problem (3)-(4) is Lyapunov uniformly local stable (or; shortly, LUS) if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$, independent on times, such that $|x^0 - \breve{x}^0|^* \le \delta$ implies $\|x - \breve{x}\|_{C^*} \le \varepsilon$, where $\breve{x} \in C^*$is the global solution of (20)-(21).

Theorem5.9 .Assume as in Theorem 4.6 that $(A_1) - (A_3)$ are satisfied globally in C^*. Then, the unique global solution of (3)-(4) is a LUS in C^*

Proof. Let, due to Theorems 3.1 and 4.6, the unique global solutions $x, \breve{x} \in C^*$ of (3)-(4) and (21). Let $x_i, \breve{x}_i \in C^*$ for any $t \in [t_0, T]$, where

$$\breve{x}_i(t) = a_i\left[\breve{x}_i^0 - \alpha_i\int_0^{\tau_i} f_i\left(s,\breve{x}_j(s)\right)ds - \beta_i\int_0^{\gamma_i} f_i\left(s,\breve{x}_j(s)\right)ds\right] + \int_0^t f_i\left(s,\breve{x}_j(s)\right)ds,$$

$$|x_j(t) - \breve{x}_j(t)| \le a_i|x_i^0 - \breve{x}_i^0| + a_i\alpha_i\int_0^{\tau_i}\left|f_i\left(s,x_j(s)\right) - f_i\left(s,\breve{x}_j(s)\right)\right| e^{-\lambda(s-t_0)}e^{\lambda(s-t_0)}ds$$

$$+\ a_i\beta_i\int_0^{\gamma_i}\left|f_i\left(s,x_j(s)\right) - f_i\left(s,\breve{x}_j(s)\right)\right| e^{-\lambda(s-t_0)}\,e^{\lambda(s-t_0)}ds$$

$$+\int_0^t\left|f_i\left(s,x_j(s)\right) - f_i\left(s,\breve{x}_j(s)\right)\right| e^{-\lambda(s-t_0)}e^{\lambda(s-t_0)}ds$$

$$|x_j(t) - \breve{x}_j(t)| \le a_i|x_i^0 - \breve{x}_i^0| + a_i\alpha_i\int_0^{\tau_i} L_i|x_j - \breve{x}_j|e^{-\lambda(s-t_0)}e^{\lambda(s-t_0)}\,ds$$

$$+\ a_i\beta_i\int_0^{\gamma_i} L_i|x_j - \breve{x}_j|e^{-\lambda(s-t_0)}e^{\lambda(s-t_0)}\,ds + \int_0^t L_i|x_j - \breve{x}_j|\,e^{-\lambda(s-t_0)}e^{\lambda(s-t_0)}ds$$

Multiplying by $e^{-\lambda(t-t_0)}$and taking the supremum when $t \in [t_0, T]$ leads to

$$\|x_j(t) - \breve{x}_j(t)\|^* \le a_i|x_i^0 - \breve{x}_i^0|^* + a_i\alpha_i\,L_i\|x_j - \breve{x}_j\|^*\int_0^{\tau_i} e^{\lambda(s-t)}ds + a_i\beta_i L_i\|x_j - \breve{x}_j\|^*\int_0^{\gamma_i} e^{\lambda(s-t)}\,ds$$

$$+\ L_i\|x_j - \breve{x}_j\|^*\int_0^t e^{\lambda(s-t)}\,ds$$

Similar to the proof's argument of Theorem 4.6, we integrating over $[t_0, T]$to conclude for $\lambda > 0$

$$\|x_j(t) - \breve{x}_j(t)\|^* \le a_i|x_i^0 - \breve{x}_i^0|^* + \frac{2}{\lambda}L_i\|x_j - \breve{x}_j\|^*$$

Set $a = \sum_{i=1}^3 a_i$, then

$$\|x - \breve{x}\|^* \le a|x^0 - \breve{x}^0|^* + \frac{2L}{\lambda}\|x - \breve{x}\|^*$$

For $\lambda > 2L$ is large enough one can estimate

$$\|x - \breve{x}\|^* \le \frac{a\lambda\delta}{\lambda - 2L} < \varepsilon\,.$$

Remark 5.10. Note that, in Theorem 5.9, we have just proved that the unique global solution $x \in C^*$ is uniformly continuous dependence on the initial data, which sufficiently exhibits the LUS property. Also, as a consequence, this property holds for all parameters and given system functions; indeed, the proof follows the same arguments used in Theorems 5.4, 5.6, and 5.9. Since the stability holds for all space data $(x_0, \alpha, f) \in D(GS)$ (the global domain of existence on$[0, T]$), it can also be considered globally Lyapunov stable (GLS) in the sense of (3)-(4). This follows from the fact that all possible small (or, large) perturbations in all space data should result in small differences between solutions at all later times, ensuring the resulting solution remains entirely within C^* (the basin of attraction).

A direct consequence of Theorems 4.6, 5.9 and Remark 5.10 is the powerful connection result.

Corollary 5.11. Suppose as in Theorem 4.6 that $(A_1) - (A_3)$ hold globally in C^*. Then, the unique global solution of (3)-(4) is a LUS in C^*if and only if it is a GLS in C^*. More precisely, $x \in C^*$ is uniformly globally stable (UGS) in the (3)-(4) sense.

5. Conclusion

In this paper, we have studied the existence and uniqueness of local and global solutions for the nonlocal problem (3)-(4) via the Banach's fixed-point theorem. The sufficient conditions for the stability of solutions has been proved by verifying local and uniform continuous dependence of local and global solutions; respectively, on the space data (i.e., the initial data, parameters, and given system functions).

Compliance with ethical standards

Disclosure of conflict of interest

The author(s) declare that they have no conflict of interest.

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