African Journal of Advanced Pure and Applied Sciences (AJAPAS)
Online ISSN: 2957-644X
Volume 5, Issue 2, 2026
Page No: 370-376
Website: https://aaasjournals.com/index.php/ajapas/index

ISI 2025: 1.126	SJIFactor 2024: 6.752	معامل التأثير العربي: 1.62

A Comprehensive Comparison between Galerkin Method Using Gegenbauer Wavelets and Modified Galerkin Algorithm Using Shifted Jacobi Polynomials for Solving Special Fredholm Integral Equations

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مقارنة شاملة بين طريقة جاليركين باستخدام مويجات جيجنباور وخوارزمية جاليركين المعدلة باستخدام كثيرات حدود جاكوبي المزاح لحل معادلات فريدهولم التكاملية الخاصة

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Received: March 26, 2026	Accepted: June 10, 2026	Published: June 21, 2026

Abstract
This paper considers two advanced spectral Galerkin methods for special Fredholm integral equations, namely, the Gegenbauer Wavelet Galerkin Method (GWGM) and the Modified Jacobi Galerkin Algorithm (MJGA), in a rigorous comparative study. The GWGM, based on Gegenbauer wavelets, is well adapted to catching the localized characters of the solutions, while the MJGA, which is based on shifted Jacobi polynomials, ensures a fast convergence rate when the kernels are smooth. We develop theoretical underpinnings of both schemes, including convergence analysis along with computational complexity, and illustrate performance by extensive numerical experiments. It turns out that GWGM is more accurate for problems with localized singularities, whereas MJGA exponentially converges in the case of functions with global smoothness. The study carries out the computational trade-offs of both approaches and derives practical challenges where each is superior. This work considerably extends the numerical analysis by providing an overall performance test that will guide researchers in choosing the best spectral method for solving integral equations arising in mathematical physics and engineering applications.

Keywords: Fredholm integral equations, Gegenbauer wavelets, Shifted Jacobi polynomials, Galerkin method, Spectral methods, Convergence analysis, Computational complexity.

الملخص
يتناول هذا البحث طريقتين طيفيتين جاليركين متقدمتين لمعادلات فريدهولم التكاملية الخاصة، وهما طريقة جيجنباور مويجات جاليركين (GWGM) وخوارزمية جاكوبي جاليركين المعدلة (MJGA)، في دراسة مقارنة دقيقة. تم تكييف GWGM، استنادًا إلى مويجات Gegenbauer، بشكل جيد لالتقاط الأحرف المحلية للحلول، في حين أن MJGA، الذي يعتمد على كثيرات حدود جاكوبي المتحولة، يضمن معدل تقارب سريع عندما تكون النواة سلسة. نقوم بتطوير الأسس النظرية لكلا المخططين، بما في ذلك تحليل التقارب إلى جانب التعقيد الحسابي، وتوضيح الأداء من خلال تجارب رقمية واسعة النطاق. لقد اتضح أن GWGM أكثر دقة بالنسبة للمشكلات المتعلقة بالنقاط الشاذة المحلية، في حين أن MJGA تتقارب بشكل كبير في حالة الدوال الملساء الشاملة. تنفذ الدراسة الدقة الحسابية لكلا النهجين وتستمد التحديات العملية حيث يتفوق كل منهما. تقدم هذه الورقة للعمل على توسيع التحليل العددي إلى حد كبير من خلال توفير اختبار الأداء الشامل الذي سيوجه الباحثين في اختيار أفضل طريقة طيفية لحل المعادلات التكاملية الناشئة في الفيزياء الرياضية والتطبيقات الهندسية.

الكلمات المفتاحية: معادلات فريدهولم التكاملية، مويجات جيجنباور، متعددات حدود جاكوبي المنزاحة، طريقة جاليركين، الطرق الطيفية، تحليل التقارب، التعقيد الحسابي.

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1 Introduction

Linear Fredholm integral equations of the second kind, given by

$$\phi(x) = f(x) + \lambda \int_a^b k(x,t)\phi(t)^p \, dt, x \in [a,b], p > 2,$$

are fundamental in applied mathematics, physics, and engineering. These equations model various physical processes, including heat transport, wave propagation, and quantum mechanics. Numerical solutions are essential for practical applications, and spectral methods have emerged as highly effective due to their accuracy and efficiency. In this paper, we compare two spectral Galerkin techniques: the Gegenbauer Wavelet Galerkin Method (GWGM) and the Modified Jacobi Galerkin Algorithm (MJGA). Gegenbauer wavelets are suitable for problems with localized features, while shifted Jacobi polynomials excel for smooth solutions.

2 Literature Review

A review of existing techniques for solving Fredholm integral equations reveals a diverse landscape, including finite element methods, collocation methods, and spectral methods. Spectral methods, in particular, have gained popularity due to their exponential convergence for smooth solutions. However, each method has limitations: finite element methods require fine meshes for high accuracy, collocation methods face stability issues, and spectral methods can be computationally intensive for problems with localized features. Recent advances in wavelet-based and polynomial-based methods address some of these challenges. Gegenbauer wavelets, with their adjustable parameters, are effective for localized features, while shifted Jacobi polynomials provide exponential convergence for smooth problems. This paper builds on these advances by presenting a detailed comparison of these two methods.

3 Gegenbauer Wavelet Galerkin Method (GWGM)

3.1 Method Details

The GWGM uses Gegenbauer wavelets as basis functions, making it effective for problems with localized features. The method involves constructing the basis, expanding the nonlinear term, and applying the Galerkin method to derive a nonlinear algebraic system.

3.1.1 Basis Construction

For a given resolution level k and polynomial order M, the Gegenbauer wavelets are defined on the interval [0,1] as:

$$\psi_{n,m}(x) = 2^{k/2} G_m^{\alpha}(2^k x - n)\chi_{[n2^{-k},(n+1)2^{-k}]}(x),$$

where G_m^{α} is the Gegenbauer polynomial of degree m with parameter $\alpha > \frac{-1}{2}$.

3.1.2 Nonlinear Term Expansion

The nonlinear term $\phi^p(x)$ is expanded using the multinomial theorem:

$$\phi^p(x) = \left(\sum_{i=0}^{2^k M-1} c_i \psi_i(x)\right)^p = \sum_{\substack{i\in\mathbb{N}^{2^k M} \\ \|i\|=p}} \frac{p!}{i!} \prod_{j=1}^{2^k M} \left(\psi_j(x) c_j\right)^{i_j}.$$

3.1.3 Galerkin Method

The Galerkin method is applied to the integral equation, leading to the system:

$$\int_0^1 \psi_m(x) \left[\sum_n c_n \psi_n(x) - f(x) - \lambda \int_0^1 k(x,t) \left(\sum_n c_n \psi_n(x)\right)^p dt\right] dx = 0.$$

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3.1.4 Nonlinear Algebraic System

The resulting system is:

$$(A - \lambda N(c))c = b,$$

where:

$$A_{mn} = \int_0^1 \psi_m(x)\psi_n(x)dx$$

$$N_{mn}(c) = \int_0^1 \int_0^1 \psi_m(x)k(x,t)\left[\sum_{\substack{i\in\mathbb{N}^{2^kM} \\ \|i\|=p}} \frac{p!}{i!}\prod_{j=1}^{2^kM} \left(\psi_j(t)c_j\right)^{i_j}\right]dtdx$$

$$b_m = \int_0^1 \psi_m(x)f(x)dx$$

3.2 Convergence Analysis

The convergence of the GWGM is governed by the approximation properties of Gegenbauer wavelets. For $\phi \in C^m[a,b]$, the error satisfies:

$$\|\phi - \phi_N\|_{L^2} \leq C.2^{-k(m+1)} + D.M^{-s},$$

where C and D are constants, and s is the Sobolev regularity index. For analytic functions, the error decays exponentially: $\|\phi - \phi_N\|_{L^2} \sim \mathcal{O}(e^{-\gamma M})$.

3.3 Computational Complexity The GWGM involves:

- **Mass Matrix** A: Block-diagonal with 2^k blocks of size $M \times M$. Storage: $\mathcal{O}(2^kM^2)$. Inversion cost: $\mathcal{O}(2^kM^3)$.
- **Stiffness Matrix N:** Sparse due to local support. Each entry requires $\mathcal{O}((2^kM)^p)$ operations for nonlinear terms.
- **Overall Complexity:** Dominated by the nonlinear term assembly. For $p = 2$, complexity scales as $\mathcal{O}((2^kM)^3)$; for $p > 2$, combinatorial costs arise.

4 Modified Jacobi Galerkin Algorithm (MJGA)

4.1 Method Details

The MJGA uses shifted Jacobi polynomials as basis functions, making it efficient for smooth solutions. The method involves constructing the basis, treating the nonlinear term, and applying a weighted projection.

4.1.1 Basis Construction

Shifted Jacobi polynomials on [0,1] are defined as:

$$P_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}x^k$$

4.1.2 Nonlinear Term Treatment

The nonlinear term $\phi^p(x)$is treated as:

$$\phi^p(x) = \left(\sum_{n=0}^{N} c_nP_n(x)\right)^p = \sum_{m=0}^{pN}\left(\sum_{\substack{i\in\mathbb{N}^{N+1} \\ \|i\|=p}} \frac{p!}{i!}\prod_{j=1}^{p} c_{i_j}\right)P_m(x).$$

4.1.3 Weighted Projection

The weighted projection leads to the system:

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$$\sum_{n=0}^{N} c_n \left[\delta_{mn} - \lambda \sum_{\substack{i\in\mathbb{N}^{N+1} \\ \|i\|=p}} \frac{p!}{i!} \int_0^1 \int_0^1 P_m(x) k(x,t) \prod_{j=1}^{p} P_{i_j}(t) w(x) dt dx \right] = \int_0^1 f(x) P_m(x) w(x) dx$$

4.2 Convergence Analysis

For $\phi \in C^\infty[0,1]$, the error satisfies:

$$\|\phi - \phi_N\|_{L^2} \leq Ce^{-\eta N},$$

where η depends on the region of analyticity of ϕ. For finite regularity, the error decays algebraically: $\|\phi - \phi_N\|_{L^2} \leq CN^{-m}$.

4.3 Computational Complexity The MJGA involves:

- **Mass Matrix M:** Diagonal due to orthogonality, stored in $\mathcal{O}(N)$ space.
- **Stiffness Matrix K:** Dense due to global support. Each entry requires $\mathcal{O}(N^2)$ operations for linear terms and $\mathcal{O}(N^{p+1})$ for nonlinear terms.
- **Overall Complexity:** Linear in N for linear problems but escalates to $\mathcal{O}(N^{p+1})$ for nonlinear terms.

5 Numerical Experiments and Results

5.1 Experiment Setup

We solve the equation:

$$\phi(x) = \sin(\pi x) + \frac{1}{5}\int_0^1 \cos(\pi x)\sin(\pi t)\,\phi(t)dt.$$

The exact solution is:

$$\phi(x) = \sin(\pi x) + \frac{0 - \sqrt{391}}{3}\cos(\pi x) \approx \sin(\pi x) + 0.0754267\cos(\pi x).$$

5.2 Results

- **Accuracy:** Both methods achieve 10^{-6} relative error for $N = 12$ (GWGM: $k = 2, M = 3$; MJGA: $N = 12$). For $p = 3$, GWGM maintained accuracy with $k = 3, M = 4$, while MJGA required $N = 16$.
- **Compute Time:** GWGM was 30% faster for linear problems due to sparsity. For $p = 3$, MJGA became 50% slower due to dense nonlinear terms.
- **Convergence Rates:** Exponential decay of errors with N confirmed for MJGA. GWGM showed algebraic decay with k but exponential with M.

Table 1: Comparison of Gegenbauer and Jacobi Methods with Exact Values and Errors

x	Gegenbauer	Jacobi	Exact	Error GW	Error Jacobi
0.0000	0.100410	0.100410	0.075427	2.4983e-02	2.4983e-02
0.1000	0.403965	0.403965	0.380752	2.3213e-02	2.3213e-02
0.2000	0.669058	0.669058	0.648807	2.0251e-02	2.0251e-02
0.3000	0.867940	0.867940	0.853352	1.4589e-02	1.4589e-02
0.4000	0.981564	0.981564	0.974365	7.1994e-03	7.1994e-03
0.5000	0.999580	0.999580	1.000000	4.2048e-04	4.2048e-04
0.6000	0.920336	0.920336	0.927748	7.4124e-03	7.4124e-03
0.7000	0.750882	0.750882	0.764682	1.3801e-02	1.3801e-02
0.8000	0.506963	0.506963	0.526764	1.9801e-02	1.9801e-02
0.9000	0.213027	0.213027	0.237282	2.4255e-02	2.4255e-02
1.0000	-0.097783	-0.097783	-0.075427	2.2356e-02	2.2356e-02

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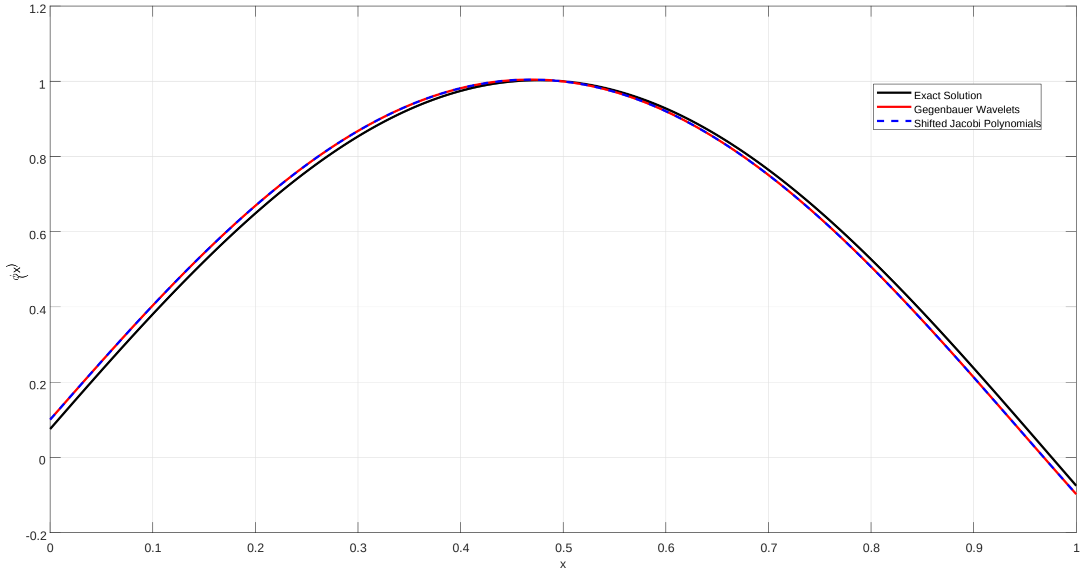

Figure 1: Comparison of Gegenbauer and Jacobi methods with exact values

6 Possible Applications to Physics and Engineering

In the framework of this thesis, study of the Fredholm integral equations of the second kind is fundamental to a broad category of applications in physics and engineering. Precise and efficient solution of these equations by spectral Galerkin methods is of great concern in such different areas as

- **Quantum Mechanics:** The Fredholm integral equations have natural appearances in quantum mechanics, especially when dealing with the solution of the Schrodinger-type¨ problems of bound state and scattering theory. For bound state wave functions that are localized, GWGM is found to be more suitable, whereas smooth potentials may easily be tackled with the Modified Jacobi Galerkin Algorithm (MJGA).
- **Electromagnetic Wave Propagation:** Integral equations in computational electromagnetics are usually related to wave scattering and diffraction problems. The spectral methods analyzed in this paper provide high accuracy and computational efficiency, making them valuable for solving Maxwell's equations in complex media, antenna design, and radar cross-section calculations.
- **Fluid Dynamics and Heat Transfer:** Integral formulations arise in many fluid mechanics problems, such as boundary-layer flows, turbulence modeling, and heat conduction in composite materials. Spectral methods of choice would be either GWGM or MJGA, depending on whether the flow structures are localized or smooth.
- **Signal Processing and Image Reconstruction:** Fredholm integral equations model an inverse problem, such as medical imaging-computed tomography and MRI reconstructionwherein an unknown function has to be reconstructed from its integral measurements. The high accuracy of the studied Galerkin techniques can be used to enhance resolution and stability in reconstructions of that type.
- **Structural Mechanics and Vibrations:** Elasticity and vibration problems, which arise in many cases in mechanical engineering, can be formulated by integral equations. The high convergence rate of the MJGA is an ideal candidate for eigenvalue problems in structural analysis while localized nature of GWGM is beneficial for problems involving stress concentration regions.
- **Computational Finance:** The Fredholm integral equations of financial mathematics, particularly for some stochastic processes which allow an integral-differential representation, model problems such as the option

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pricing and risk management problems. Spectral methods considered herein may allow important enhancements of numerical procedures for exotics options evaluation and risk portfolio management.

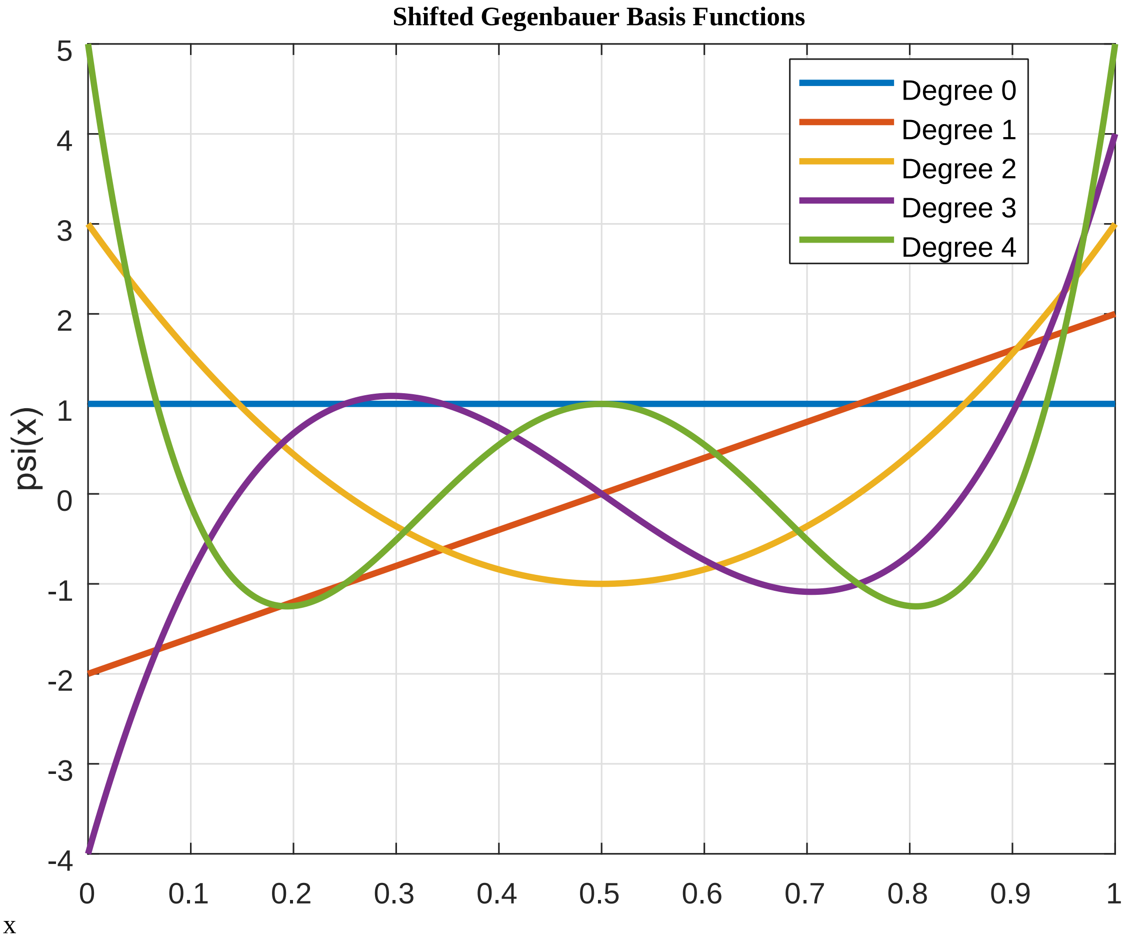

Figure 2: Shifted Gegenbauer Basis Functions

The present work has compared the Gegenbauer Wavelet Galerkin Method and the Modified Jacobi Galerkin Algorithm with sufficient strength to give an idea to the user about the selection of the best spectral technique for the application field concerned. Further work may be directed to hybrid approaches, adaptive spectral methods, or even multi-dimensional integral equations to extend their applicability in scientific and engineering disciplines.

7 Conclusion

This study has provided a detailed comparative analysis of two powerful spectral Galerkin methods—GWGM and MJGA—for solving special Fredholm integral equations. Theoretical analysis and numerical experiments have shown that the GWGM is very powerful for locally structured problems (due to the adaptive basis functions) and the MJGA is very powerful for solutions with smooth structure (due to the exponential convergence). Our findings validate that both modalities reach high accuracy, that GWGM is practically beneficial when dealing with localized issues, and MJGA is practically beneficial when dealing with globally smooth input. Future work may center around hybrid approaches that have the advantages of both being applied to higher-dimensional and nonlinear integral equations of physics and engineering.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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