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Application of Adomian Decomposition Method Solving Volterra Integral Equation

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تطبيق طريقة تفكيك أدوميان لحل معادلة فولتيرا التكاملية

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Abstract
In this work, definitions and classifications of integral equations are described. Next, Adomian decomposition method is observed. Moreover, Adomian decomposition method is set out with the aim of solving linear and nonlinear Volterra integral equations. Finally, some applications are expressed to demonstrate the effectiveness of Adomian decomposition methods for solving linear and nonlinear Volterra integral equations, the modified decomposition method and the Laplace transform method by using MATLAB.

Keywords: Adomian Decomposition Method, Exact Solution, MATLAB, series solution, Volterra Integral Equation.

الملخص
يتناول هذا العمل تعريفات وتصنيفات المعادلات التكاملية. ثم يُناقش أسلوب تفكيك أدوميان. علاوة على ذلك، يُشرح أسلوب تفكيك أدوميان بهدف حل معادلات فولتيرا التكاملية الخطية وغير الخطية. وأخيرًا، تُعرض بعض التطبيقات لتوضيح فعالية أساليب تفكيك أدوميان في حل معادلات فولتيرا التكاملية الخطية وغير الخطية، بالإضافة إلى أسلوب التفكيك المُعدَّل وأسلوب تحويل لابلاس باستخدام برنامج MATLAB.

الكلمات المفتاحية: أسلوب تفكيك أدوميان، الحل الدقيق، MATLAB، حل المتسلسلات، معادلة فولتيرا التكاملية.

Introduction.
Integral equations play an important role in mathematical modeling by providing a powerful framework for describing relationship between functions in terms of integral. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations.
Several scientific and engineering applications are usually described by integral
equations or integro differential equations. Integral equations arise in the potential theory more than any other field, also in diffraction problems, conformal mapping, water waves, scattering in quantum mechanics, and Volterra's population growth model. The electrostatic, electromagnetic scattering problems and propagation of acoustical and elastically waves are scientific fields where integral equations appear.
Integral equations are also important in the theory and numerical analysis of differential equations; this is where the mathematic student is most likely to encounter them. For example, Picard's existence and uniqueness theorem for first-order initial value problems is conveniently proved using integral equations. We give an introductory

overview of Volterra integral equation, and some traditional and new techniques for solve it. Furthermore, we use MATLAB for some techniques.

Definition 1.1:[1]. An integral equation is one in which the unknow function $u(x)$ occurs within an integral sign. The most common sort of integral equation in $u(x)$ is of the form

$$h(x)u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x-t)u(t)dt\,, \qquad x \in [a,b] \qquad (1.1)$$

Where $\alpha(x)$ and $\beta(x)$ are the limits of integration, might be variables, constants or mixed, and they can be in one dimension or two or more. λ is a constant parameter, and $k(x-t)$ is a known function of two the variables$x$ and $t,$ called the Kernel of the integral equation. The functions $f(x)$ and $k(x-t)$ are given in advance.

Example 1.1:[1,2]

$$u(x) = \int_0^1 (x-t)u(t)dt \qquad (1.2)$$

$$u(x) = 1 + \frac{1}{4}\int_0^x xu(t)dt \qquad (1.3)$$

1.1 Classification of Integral Equations:[2]

An Integral equations appear in many types. The types depend on the linearity, homogeneity, the kernel of the equation and kind of integral equation. The most frequently used integral equations fall under two major classes, namely volterra and fredholm integral equations. Of course, we have to classify them as problems, we come a cross singular equations.

In this text, we shall distinguish four major types of integral equation-the two main classes and two related types of integral equations. In particular, the four types are given below.

- Fredholm integral equations
- Volterra integral equations
- Integro-differential equations
- Singular integral equations

In this article the researcher will work on the only first and second types. We shall outline these equations using basic definitions and properties of each type.

1- Fredholm integral equation:[1,2]

Definition: An integral equation (1.1) is called Fredholm integral equation, when $\alpha(x) = a, \beta(x) = b$,where a and b are constants.The general form is:

$$h(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \qquad x \in [a,b] \qquad (1.4)$$

Example 1.1:[1]

$$u(x) = e^x + \int_0^2 xu(t)dt = 0 \qquad (1.5)$$

$$u(x) = \frac{9}{10}x^2 + \int_0^1 \frac{1}{2}x^2t^2u(t)dt \qquad (1.6)$$

The equation (1.3) is fredholm integral equation of the first kind, and an equation (1.4) is fredholm integral equation of the second kind.

2-Volterra integral equation:[1,3]

Definition: In Volterra integral equation (1.1), at least one of the limits of integration is a variable.

For the first kind Volterra integral equation, the unknown function $u(x)$ appears only inside integral sign in the form:

$$f(x) = \int_0^x k(x,t)u(t)dt \qquad (1.7)$$

Example 1.2:[1]

$$u(x) = 1 - \int_0^x (x-t)u(t)dt \qquad (1.8)$$

$$u(x) = 1 - \int_0^x 1 + (x-t)u^4(t)dt \qquad (1.9)$$

Main Results

Volterra integral equations appear in a variety epidemic spread.Volterra started working on integral equations in 1884, but his serious study began in 1896. The term Volterra integral equation was first intoduced by Lalesco in 1908. Volterra integral equations, of the first kind or the second kind, are defind by a variable upper limit of integration.

$$h(x)u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt, \tag{1.10}$$

The kernel $k(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter.

Example:

$$u(x) = 1 - \frac{1}{3}x^2 + \int_0^x tu(t)dt \tag{1.11}$$

In the subsequent sections, we will present a variety of analytic and numerical techniques, new and traditional, that will be used in studying Volterra integral equations.

In this document we utilize newly developed approaches, specifically, the Adomian decomposition method (ADM), the modified decomposition method (mADM), and the variational iteration method (VIM) to handle Volterra integral equations.

Some of the traditional approaches, specifically, successive approximations method, and the Laplace transform method will be employed as well.

2.2 The Adomian Decomposition Method (ADM):[1,2,4,12,15]

The Adomian decomposition approach (ADM) seems effective for both linear and nonlinear differential equations,as well as integral equations, and integro-differential equations. This technique was publications by George Adomian in early 1990 in his books [1] and [2]. The Adomian decomposition approach involves breaking down the unknown function $u(x)$ of any equation into an infinite series.

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{1.12}$$

with $u_0(x)$ as the term outside the integral sign, i.e.

$$u_0(x) = f(x), \tag{1.13}$$

The decomposition method concerns itself with finding the components $u_0, u_1, u_2, \ldots$ individually.

Substituting (1.12) into (1.10) yields

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt, \tag{1.14}$$

Or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_0^x k(x,t)[u_0(t) + u_1(t) + \cdots]dt, \tag{1.15}$$

The components $u_i(x),\ i \geq 0$ of the unknown function $u(x)$ are completely determind by using the recurrenve,

$$\begin{aligned}
u_0(x) &= f(x),\\
u_1(x) &= \lambda \int_0^x k(x,t)\, u_0 dt,\\
u_2(x) &= \lambda \int_0^x k(x,t) u_1 dt\\
&.\\
&.\\
&.\\
u_n(x) &= \lambda \int_0^x k(x,t) u_{n-1}(t)dt\,, \qquad n \geq 1
\end{aligned} \tag{1.16}$$

This set of equations (1.16) can be written in compact recurrence relation as

$$u_0(x) = f(x),$$

$$u_{n+1}(x) = \lambda \int_0^x k(x,t)u_n(t)dt, \quad n \geq 0 \tag{1.17}$$

It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. We are using MATLAB for numerical solution **[A.1]**.

Example:[1,5] the Volterra integral equation by using ADM.

$$u(x) = x + \int_0^x (t-x)u(t)dt \tag{1.18}$$

We have $f(x) = x,\ \lambda = 1\,,\ k(x-t) = t - x,$ substituting (1.12) into both sides of (1.18), we get

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x (t-x)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt \tag{1.19}$$

Now decomposing the different terms in the following manner, we get set of solutions

$$u_0(x) = x, \tag{1.20}$$

$$u_1(x) = \int_0^x (t-x)\, u_0(t)dt$$

$$= \int_0^x (t-x)tu(t)dt = \left[\frac{t^3}{3}\right]_0^x - x\left[\frac{t^2}{2}\right]_0^x = \frac{-x^3}{6} = -\frac{x^3}{3!} \tag{1.21}$$

$$u_2(x) = \int_0^x (t-x)\, u_1(t)dt$$

$$= \int_0^x (t-x)\left(-\frac{1}{3!}\,t^3\right)dt$$
$$= -\frac{1}{3!}\left(\left[\frac{t^5}{5}\right]_0^x - \left[\frac{t^4}{4}\right]_0^x\right) = -\frac{1}{6}\left(-\frac{x^5}{20}\right) = \frac{x^5}{5!} \qquad (1.22)$$
$$u_3(x) = \int_0^x (t-x)u_2(t)dt$$
$$= \int_0^x (t-x)\left(\frac{1}{5!}t^5\right)dt$$
$$= \frac{1}{5!}\left(\left[\frac{t^7}{7}\right]_0^x - x\left[\frac{t^6}{6}\right]_0^x\right)$$
$$= \frac{1}{5!}\left[\frac{-x^7}{42}\right] = \frac{-x^7}{7!} \qquad (1.23)$$
And so on.Using (1.12) gives the series solution:
$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \qquad (1.24)$$
That converges to the closed form solution:
$$u(x) = sinx$$

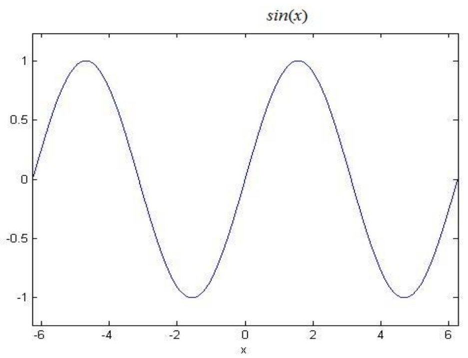

Figure 1. Graph of each component solutions

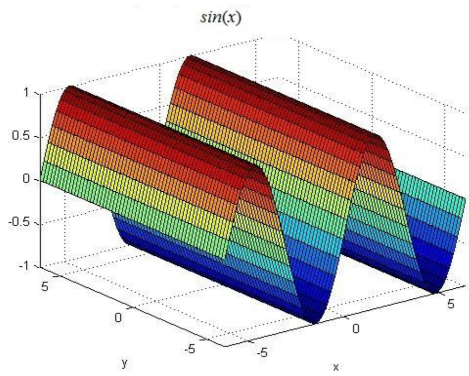

Figure 2. Graph of the exact solutions

2.3 The Modified Decomposition Method (mADM):[4,5,10,13]

It is worth noting that the Adomian decomposition method was developed by Wazwaz and presented in [10]. The (mADM) is appropriate to all integral equations and differential equations of any order. It is depends mainly on splitting the function $f(x)$ into two parts, therefore it cannot be used if the function $f(x)$ consists of only one term.

The modified decomposition method will be applied in the Volterra integral equation (1.10) if the function $f(x)$ consists of a combination of two or more of polynomials , trigonometric functions, hyperbolic functions, and others.

We decompose the function $f(x)$ into the sum of two partial functions, such as
$$f(x) = f_0(x) + f_1(x) \qquad (1.25)$$
where $f_0(x)$ consists of unmber of terms of $f(x)$ and $f_1(x)$, remaining terms of $f(x)$. Accordingly (1.10), we get
$$u(x) = f_0(x) + f_1(x) + \lambda \int_0^x k(x,t)\ u(t)dt \qquad (1.26)$$
Substituting (1.12) into (1.25), we obtatin
$$u_0(x) + u_1(x) + u_2(x) + \cdots = f_0(x) + f_1(x) + \lambda \int_0^x k(x,t)u(t)[u_0(t) + u_1(t)]dt \quad (1.27)$$
Consequently, the components $u_i(x), i \geq 0$ of the unknown function $u(x)$ can be completely determined in a modified recurrence relation as the following
$$u_0(x) = f_0(x),$$
$$u_1(x) = f_1(x) + \lambda \int_0^x k(x,t)u(t)dt$$
$$u_2(x) = \lambda \int_0^x k(x,t)u(t)dt,$$
$$u_3(x) = \lambda \int_0^x k(x,t)u(t)dt,$$
.
$$(1.28)$$
.

.

$u_{n+1}(x) = \lambda \int_0^x k(x,t)u(t)dt\,, \qquad n \geq 1$

The set of equation (1.28) can be written in compact recurrence relation as:

$u_0(x) = f_0(x),$

$u_1(x) = f_1(x) + \lambda \int_0^x k(x,t)u_0(t)dt$ (1.29) $u_{n+1}(x) = \lambda \int_0^x k(x,t)u(t)dt\,, \qquad n \geq \ 1$

The success of this modification depends only on the proper of and. A rule that may help for the proper choice of and could not be found yet. We are using MATLAB for numerical solution **[A.2]**.

Example: The Volterra integral equation by using the modified decomposition method

$$u(x) = 2x - \left(1 - e^{-x^2}\right) + \int_0^x e^{-x^2+t^2} u(t)dt \qquad (1.30)$$

We first decompose the function $f(x)$ into

$$f_0(x) = u_0(x) = 2x \qquad (1.31)$$

and

$$f_1(x) = -(1 - e^{-x^2})$$

$$u_1(x) = -\left(1 - e^{-x^2}\right) + \int_0^x e^{-x^2+t^2}\, u_0(t)dt \qquad (1.32)$$

$$= -\left(1 - e^{-x^2}\right) + e^{-x^2} \int_0^x e^{t^2}\, 2t\ \ dt$$

$$= -\left(1 - e^{-x^2}\right) + e^{-x^2}\left[e^{t^2}\right]_0^x$$

$$= -\left(1 - e^{x^2}\right) + e^{-x^2}(e^{x^2} - 1)$$

$$= -\left(1 - e^{-x^2}\right) + \left(1 - e^{-x^2}\right) = 0$$

Accordingly, other components $u_i(x) = 0$, for $i \geq 2$. Then the exact solution is

$$u(x) = 2x.$$

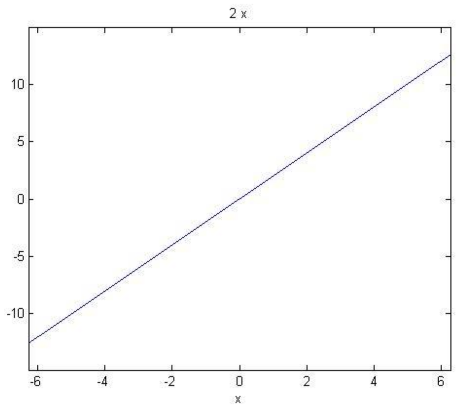

Figure 3. Graph of each compoent solutions

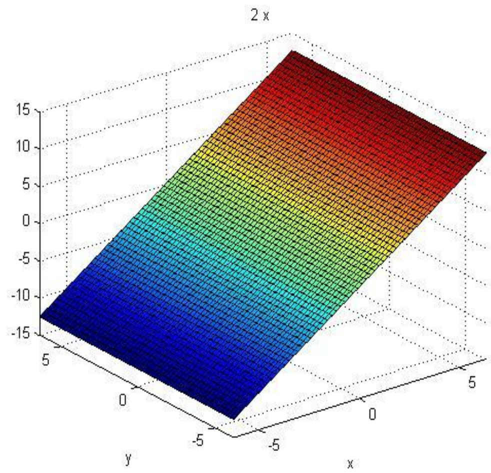

Figure 4. Graph of the exact solutions

2.4 The Varialtional Iteration Method (VIM): [6,7,9,8,11,14]

In this section we will study the newly developed variational iteration method that proved to be effective and reliable for analytic and numerical purposes.The variational iteration method (VIM) established by Ji-Huan He [8-11] is now used to handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations.

$$L_u + N_u = g(t), \tag{1.33}$$

Where L and N are linear and nonlinear operators respectively, and $g(t)$ is inhomogeneous term. The variational iteration method presents a correction functional for equation (1.33),in the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\varepsilon)\left(Lu_n(\varepsilon) + N\tilde{u}_n(\varepsilon) - g(\varepsilon)\right)d\varepsilon \tag{1.34}$$

Where λ is a general Lagrange's multiplier, noting that in this method λ may be a constant or a function, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n = 0$, where δ is the variational derivative. Having determined the Lagrange multiplier $\lambda(\varepsilon)$, the successive approximations $u_{n+1}, n \geq 0$,of the solution $u(x)$ will be readily obtained upon using selective function $u_0(x)$. However, for fast convergence, the function $u_0(x)$ should be selected by using the initial conditions as follows:

$u_0(x) = u(x)$ for first order u'_n,

$u_0(x) = u(0) + xu'(0)$ for second order u''_n,

$u_0(x) = u(0) + xu'(0) + \frac{1}{2!}x^2u''(0)$, for third order u'''_n, (1.35)

and so on. Conequently, the solution

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{1.36}$$

The determination of the Lagrange multiplier plays a major role in the determination of the solution of the problem. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multipliers, and its correction functional respectively:

$$\begin{cases} u' + f(u(\varepsilon), u'(\varepsilon) = 0, & \lambda = -1 \\ u_{n+1} = u_n - \int_0^x [u'_n + f(u'_n + f(u_n, u'_n)]d\varepsilon, \end{cases}$$

$$\begin{cases} u'' + f\left(u(\varepsilon), u'(\varepsilon), u''(\varepsilon)\right) = 0, & \lambda = (\varepsilon - x) \\ u_{n+1} = u_n + \displaystyle\int_0^x (\varepsilon - x)[u''_n + f(u_n, u'_n, u''_n)]d\varepsilon, \end{cases}$$

$$\begin{cases} u''' + f\left(u(\varepsilon), u'(\varepsilon), u''(\varepsilon), u'''(\varepsilon)\right) = 0, & \lambda = -\dfrac{1}{2!}\ (\varepsilon - x^2) \\ u_{n+1} = u_n - \displaystyle\int_0^x \frac{1}{2!}(\varepsilon - x)^2[u'''_n + f(u_n, \dots, u'''_n)]d\varepsilon \end{cases}$$

$$\begin{cases} u^{(iv)} + f\left(u(\varepsilon), u'(\varepsilon), u''(\varepsilon), u'''(\varepsilon)\right), u^{(iv)}(\varepsilon)) = 0, & \lambda = -\frac{1}{3!}(\varepsilon - x)^3 \\ u_{n+1} = u_n + \int_0^x \frac{1}{3!}(\varepsilon - x)^3\left[u'''_n + f\left(u_n, u'_n, \dots, u_n{}^{(iv)}\right)\right]d\varepsilon, \end{cases}$$

and generally

$$\begin{cases} u^{(n)} + f\left(u(\varepsilon), u'(\varepsilon), \dots, u^{(n)}(\varepsilon)\right) = 0, & \lambda = (-1)^n \dfrac{1}{(n-1)!}(\varepsilon - x)^{(n-1)} \\ u_{n+1} = u_n(-1)^n \displaystyle\int_0^x \frac{1}{(n-1)!}(\varepsilon - x)\left[u'''_n + f\left(u'''_n, \dots, u_n{}^{(n)}\right)\right]d\varepsilon \end{cases}$$

for $n \geq 1$.

To use the variational iteration method for solving Volterra integral equations, it is necessary to convert the integral equation to an equivalent initial value problem or to an equivalent integro-differential equation.

Example: The Volterra integral equation by using the variational iteration method

$$u(x) = 1 - x + \int_0^x (x-t)u(t)dt, \tag{1.37}$$

Differentiation both sides of (1.37), using Leibniz rule, we get,

$$u'(x) = -1 + \int_0^x u(t)dt \tag{1.38}$$

The initial condition $u(0) = 1$ is obtained by using $x = 0$ into (1.36). The correction functional for equation (1.38) is

$$u_{n+1}(x) = u_n(x) - \int_0^x (u'_n(\varepsilon) + 1 - \int_0^\varepsilon (u_n(\varepsilon)dr)d\,\varepsilon \tag{1.39}$$

where we selected $\lambda = -1$ for (1.39), and $u_0(x) = 1$. Using this selection into (1.39) given the following successive approximations:

$$u_0(x) = 1, \tag{1.40}$$

$$\begin{aligned}
u_1(x) &= 1 - \int_0^x \left(u'_0(\varepsilon) + 1 - \int_0^\varepsilon u_0\,(r)dr)\right)d\varepsilon, \\
&= 1 - \int_0^x 0d\varepsilon - \int_0^x d\varepsilon + \int_0^x \int_0^\varepsilon dr\; d\varepsilon, \\
&= 1 - [\varepsilon]_0^x + \int_0^x [r]_0^\varepsilon d\varepsilon \\
&= 1 - x + \left[\frac{\varepsilon^2}{2}\right]_0^x = 1 - x + \frac{1}{2!}x^2
\end{aligned} \tag{1.41}$$

$$\begin{aligned}
u_2(x) &= 1 - x + \frac{x^2}{2!} - \int_0^x \big(u'_1(\varepsilon) + 1 - \int_0^x u_1(r)dr)\big)\,d\varepsilon, \\
&= 1 - x + \frac{x^2}{2!} - \int_0^x (-1+\varepsilon)d\varepsilon - \int_0^\varepsilon d\varepsilon + \int_0^x \int_0^\varepsilon \left[(1 - r + \frac{r^2}{2!})dr\right]d\varepsilon, \\
&= 1 - x + \frac{x^2}{2!} + [\varepsilon]_0^x - \left[\frac{\varepsilon^2}{2}\right]_0^x - [\varepsilon]_0^x + \int_0^x \left[r - \frac{r^2}{2} + \frac{r^3}{3}\right]d\varepsilon, \\
&= 1 - x + \frac{x^2}{2!} + x - \frac{x^2}{2} - x + \int_0^x \left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3}\right)d\varepsilon, \\
&= 1 - x + \left[\frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} + \frac{\varepsilon^4}{24}\right]_0^x, \\
&= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}
\end{aligned} \tag{1.42}$$

recall that

$$\begin{aligned}
u(x) &= \lim_{n\to\infty} u_n(x), \\
&= \lim_{n\to\infty}\left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + (-1)^n\frac{1}{n!}x^n\right)
\end{aligned} \tag{1.43}$$

That gives the exact solution by $u(x) = e^{-x}$.

2.5 The Successive Approximation Method:[3,8,9,11]

The successive approximations method, also Called the Picard iteration method that can be used for solving initial value problems or integral equations.

This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation (is any selective real-valued function that will be used in recurrence relation to determine the other approximations). The successive approximation method of (1.12) introduces the recurrence relation.

$$u_n(x) = f(x) + \lambda \int_0^x k(x,t)u_{n-1}dt, \qquad n \geq 1 \tag{1.44}$$

However ,we always start with $u_0(x)$ any selective real-valued function (0,1 or X) and by using (1.44), several successive approximations $u_k, k \geq 1$ will be determined as

$$\begin{aligned}
u_1(x) &= f_1(x) + \lambda \int_0^x k(x,t)u_0(t)dt \\
u_2(x) &= f(x) + \lambda \int_0^x k(x,t)u_1(t)dt, \\
u_3(x) &= f(x) + \lambda \int_0^x k(x,t)u_2(t)dt, \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
u_n(x) &= f(x) + \lambda \int_0^x k(x,t)u_{n-1}dt
\end{aligned} \tag{1.45}$$

Consequently, the solution of equation (1.12) is obtained by

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{1.46}$$

So that the resulting solution $u(x)$ is indepedent of the choice of the zeroth approximation $u_0(x)$.

Example

The Volterra integral equation by using the successive approximations method

$$u(x) = 1 + 2x + 4\int_0^x (x-t)u(t)dt \tag{1.47}$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 0 \tag{1.48}$$

The method of successive approximatons admits the use of the iteration formula

$$u_{n+1}(x) = 1 + 2x + 4\int_0^x (x-t)u_n(t)dt, \quad n \geq 0 \tag{1.49}$$

Substituting (1.47) into (1.48), we obtain

$$u_1(x) = 1 + 2x + 4\int_0^x (x-t)(0)dt, \tag{1.50}$$
$$= 1 + 2x$$

$$u_2(x) = 1 + 2x + 4\int_0^x (x-t)(1+2t)dt \tag{1.51}$$
$$= 1 + 2x + 4\int_0^x (x + 2xt - t - 2t^2)dt,$$
$$= 1 + 2x + 4\left[xt + xt^2 - \frac{t^2}{2} - \frac{2}{3}t^3\right]_0^x$$
$$= 1 + 2x + 4\left(x^2 + x^3 - \frac{x^2}{2} - \frac{2}{3}x^3\right),$$
$$= 1 + 2x + 4x^2 + 4x^3 - 2x^2 - \frac{8}{3}x^3,$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

The solution $u(x)$ of (1.51) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = e^{2x}$$

2.6 The Laplace Transform Method:[1,8,9,10]

In this section we will review only the basic concepts of the Laplace transfor method. The details can be found in any text of ordinary differential equations.The Laplace transform method`is a powerful tool used for solving differential and integral equations. It was stated that if the kernel $k(x,t)$ of the integral equation (1.12) depends on the difference $x-t$, then it is called a difference kernel. Examples of the difference kernel are e^{x-t}, $\cos(x-t)$ and $x-t$

The integral equation can thus be expressed as

$$u(x) = f(x) + \lambda\int_0^x k(x-t)dt \tag{1.52}$$

Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ given by

$$\ell\{f_1(x)\} = F_1(s),$$
$$\ell\{f_2(x)\} = F_2(s), \tag{1.53}$$

The Laplace convolution product of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt, \tag{1.54}$$

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt, \tag{1.55}$$

$$(f_1 * f_2)(x) = (f_2 * f_1)(x) \tag{1.56}$$

The Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\ell\{(f_1 * f_2)(x)\} = \ell\left\{\int_0^x f_1(x-t)f_2(t)dt\right\}. = F_1(s)F_2(s) \tag{1.57}$$

By thking Laplace transform of both sides of (1.57).We find

$$U(s) = F(s) + \lambda k(s)\, U(s), \tag{1.58}$$

Where

$U(s) = \ell\{u(x)\}, \quad k(s) = \ell\{k(x)\}, \quad F(s) = \ell\{f(x)\}$ (1.59)

Solving (1.58) for $U(s)$ gives

$$U(s) = \frac{F(s)}{1-\lambda k(s)}, \qquad \lambda k(s) \neq 1 \qquad (1.60)$$

The solution $u(x)$ is obtained by taking the inverse Laplace transform of both sides of (1.60) where we find

$$u(x) = \ell^{-1}\left\{\frac{F(s)}{1-\lambda k(s)}\right\} \qquad (1.61)$$

The right side of (1.61) can be evaluated by using Table **[A.3]**. We are using MATLAB for numerical solution **[A.4]**

Example

The Volterra integral equation by using the Laplace transform method

$u(x) = 1 - \int_0^x (x-t)\, u(t)dt$ (1.62)

The kernel $k(t-x) = (x-t),\ \lambda = -1$. Taking Laplace transform of both sides (1.61) gives

$\ell\{(ux)\} = \ell\{1\} - \ell\{(x-t) * u(x)\},$ (1.63)

so that

$U(s) = \frac{1}{s} - \frac{1}{s^2}U(s),$ (1.64)

Or equivalently

$$U(s) + \frac{1}{s^2}U(s) = \frac{1}{s},$$

$$U(s) = \frac{\frac{1}{s}}{1+\frac{1}{s^2}} = \frac{\frac{1}{s}}{\frac{s^2+1}{s^2}},$$

$U(s) = \frac{s}{s^2+1}$ (1.65)

By taking the inverse Laplace transform of both sides of (1.65), the exact solution is given by

$$u(x) = cosx$$

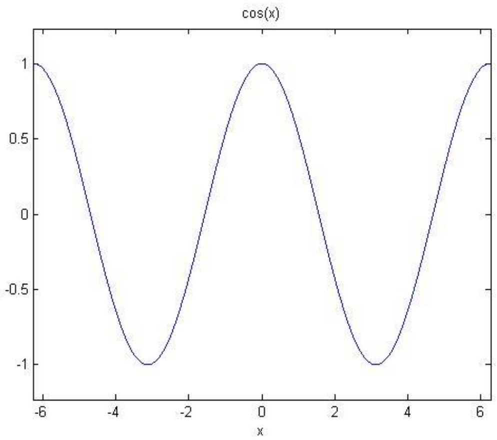

Figure 5. Graph of each component solutions

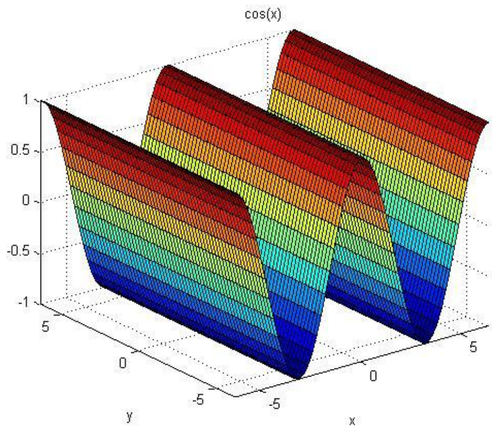

Figure 6. Graph of the exact solutions

2.7 Comparison between Alternative Methods:[13,16,17]

Now we make a comparison between all methods discussed in this chapter. However, when it comes to selecting a preferable method among the methods that were introduced in the previous section, we cannot recommend a specific method.

However, we found that if the kernel $k(x,t)$ of the integral equation is a degenerate one that consists of a polynomial of one or two terms, the decomposition method might be the best choices because it minimize the volume of calculations.
Comparing the Adomian decomposition method with the successive approximation method, it is evident the decomposition method is much easier in that we integrate always very few terms to obtain the successive components. Moreover, for linear Volterra integral equations, the Variational iteration method works effective as the Adomian method.
To achieve our goal of the comparison between these methods, we illustrate this comparison by solving the following Volterra integral equation by using all various methods.

Example 2.7.1

Solve the following by using the five alternative methods
$u(x) = 1 + \int_0^x u(t)\, dt$ (1.66)
Solution

a) Adomian Decomposition Method:
We have $f(x) = 1,\ \lambda = 1,\ k(x,t) = 1$. Substituting (1.12) into both sides of (1.66), we get
$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \left(\sum_{n=0}^{\infty} u(t)\right) dt \quad (1.67)$$
Now decomposing the different terms in the followingt manner, we get a set of solutions
$u_0(x) = 1,$ (1.68)
$u_1(x) = \int_0^x u_0(t)\, dt,$ (1.69)
$= \int_0^x dt = [t]_0^x = x$
$u_2(x) = \int_0^x u_1(t)\, dt,$ (1.70)
$= \int_0^x t\, dt = \frac{1}{2}[t]_0^x = \frac{1}{2!}x^2$
$u_3(x) = \int_0^x u_2(t)\, dt,$ (1.71)
$= \frac{1}{2}\int_0^x t^2 dt = \frac{1}{6}[t^3]_0^x = \frac{1}{3!}x^3$
and so on. Using (1.12) gives the series solution:
$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$ (1.72)
That converges to the closed forms solution:
$$u(x) = e^x$$
b) Modified Decomposition Method:
We first decomposition Method $f(x)$ into
$f_0(x) = 1$ and $f_1(x) = 1,$ (1.73)
We next use the modified recurrence firmula (1.28) to obtain
$u_0(x) = f(x) = 0,$ (1.74)
$u_1(x) = \int_0^x u_0(t)\, dt,$
$= \int_0^x dt = [t]_0^x = x$
$u_2(x) = \int_0^x u_1(t)\, dt,$ (1.75)
$= \int_0^x t\, dt = \frac{1}{2}[t^2]_0^x = \frac{1}{2!}x^2$
$u_3(x) = \int_0^x u_2(t)\, dt,$ (1.76)
$= \frac{1}{2}\int_0^x t^2 dt = \frac{1}{6}[t^3]_0^x = \frac{1}{3!}x^3$
and so on. Then the exact solution is
$$u(x) = e^x$$
c) Variational Alteration Method:
Differentiating both sides of (1.66), using Leibniz rule, we get
$u'(x) = u(x)$ (1.77)
The initial contition $u(0) = 1$ is obtained by using $x = 0$ into (1.66), the correction functional for equation (1.77)
$$u_{n+1}(x) = u_1(x) - \int_0^x (u'_n(\varepsilon) - u_n(\varepsilon) d\varepsilon, \quad n \geq 0$$
$u_0(x) = 1,$ (1.78)
$u_1(x) = 1 - \int_0^x (u'_1(t) - u_1(t)))dt,$ (1.79)

$$= 1 - \int_0^x 0dt - \int_0^x dt$$
$$= 1 + [t]_0^x = 1 + x.$$
$$u_2(x) = 1 + x - \int_0^x (u'_1(t) - u_1(t)))dt, \tag{1.80}$$
$$= 1 + x - \int_0^x dt - \int_0^x (1+t)dt,$$
$$= 1 + x - [t]_0^x + \left[t + \frac{1}{2}t^2\right]_0^x,$$
$$= 1 + x + \frac{1}{2}x^2$$
$$u_3(x) = 1 + x + \frac{1}{2}x^2 - \int_0^x (u'_2(t) - u_2(t)))dt, \tag{1.81}$$
$$= 1 + x + \frac{1}{2}x^2 - \int_0^x (1+t) - 1 + t + \frac{1}{2}t^2)dt,$$
$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3}x^3.$$
recall that
$$u(x) = \lim_{n\to\infty} u_n(x),$$
$$= \lim_{n\to\infty} \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n\right). \tag{1.82}$$
that gives the exact solution by $u(x) = e^x$.

d) Successive Approximation Method:

For zeroth approximation $u_0(x)$, we select
$$u_0(x) = 1 \tag{1.83}$$
The method of successive approximation admits the use of the iteration formule.
$$u_1(x) = 1 + \int_0^x u_0(t)\, dt, \tag{1.84}$$
$$= 1 + \int_0^x dt = [t]_0^x = 1 + x$$
$$u_2(x) = 1 + \int_0^x u_1(t)\, dt, \tag{1.85}$$
$$= 1 + \int_0^x (1+t)dt = 1 + \left[t + \frac{1}{2}t^2\right]_0^x = 1 + x + \frac{1}{2!}x^2$$
$$u_3(x) = 1 + \int_0^x u_2(t)\, dt, \tag{1.86}$$
$$= 1 + \int_0^x t^2 dt = \frac{1}{6}[t^3]_0^x = \frac{1}{3!}x^3$$
and so on.The soution of (1.66) is given by
$$u(x) = \lim_{n\to\infty} u_n(x) = e^x$$

e) Laplace transform Method:

We have $k(x-t) = 1,\ \lambda = 1$. Taking Laplace transform of both (1.66) given

$$\ell\{u(x) = \ell\{1\} + \ell\{1 * u(x)\}. \tag{1.87}$$
so that
$U(s) = \frac{1}{s} + \frac{1}{s}U(s)$, or equivalently
$$U(s) - \frac{1}{s}U(s) = \frac{1}{s},$$
$$U(s)\left(\frac{s-1}{s}\right) = \frac{1}{s},$$
$$U(s) = \frac{\frac{1}{s}}{\frac{s-1}{s}} = \left(\frac{1}{s}\right)\left(\frac{s}{s-1}\right),$$
$$U(s) = \frac{1}{s-1} \tag{1.88}$$
By taking inverse Laplace transform of both sides (1.88), the exact solution is given by
$$u(x) = e^x$$

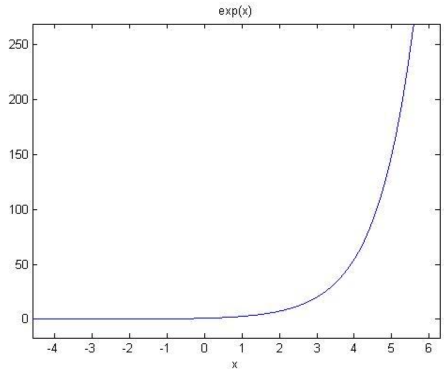

Figure 7. Graph of each component solutions

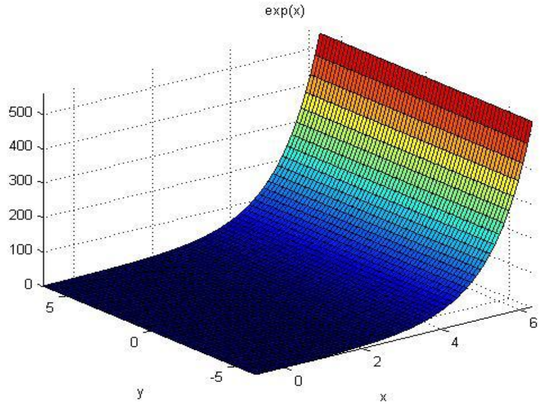

Figure 8. Graph of the exact soultions

Conclusion

In this research paper, Adomian decomposition method is applied widely in Applied Mathematics and in the field of series solutions. This method is a useful method for solving linear and nonlinear Volterra integral equations. Volterra integral equations are essential for modeling time-dependent processes with past events. This method can be applied to solve the problems in a straightforward and simple manner without changing the physical behavior of the model. The role of the kernel function is clearly seen for solving these equations. MATLAB is used to distinguish the solution of each component and the exact solutions. This method can reduce the size of the computational work and spending time. Series solutions from each component converge to the exact solution by using Adomian decomposition method. Hence, using this method it can help us to solve the linear and nonlinear Volterra integral equation easily and simply.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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