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Carathéodory Solutions for a Nonlinear Coupled Two-Dimensional Implicit Differential System with Nonlocal Boundary Constraints

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حلول كاراثيودوري لنظام تفاضلي ضمني ثنائي الأبعاد مقترن وغير خطي مع شروط حدودية غير محلية

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Abstract:
We study the existence and uniqueness of solutions for a system consisting of two coupled nonlinear first-order differential equations involving implicit derivatives. The study focuses on systems described under multipoint nonlocal boundary conditions in two-dimensional, bounded and convex domains. Using an appropriate transformation, such implicit differential system is reduced to a corresponding integral form to analyze its fixed points in a product Lebesgue space. We follow closely Schauder's approach employed to establish a rigorous existence result; in particular, we verify the compactness of the operator via the Kolmogorov compactness criterion. Uniqueness of solutions is further investigated by adapting certain conditions due to Banach contraction technique. More notably, we prove that despite the derivatives being defined in the integrable space, the solutions exist in the product space of absolutely continuous functions. The relaxed problem involving nonlocal integral conditions of the coupled system type is also successfully treated by the same framework. Finally, numerical simulations are performed in MATLAB to support our theoretical findings.

Keywords: Coupled nonlinear system, Nonlocal condition, Schauder's fixed-point theorem, Existence and uniqueness of solution, Kolmogorov compactness criterion, MATLAB.

الملخص:
تبحث هذه الورقة في وجود ووحدانية الحلول لنظام مترابط مكون من معادلتين تفاضليتين غير خطيتين ومن الرتبة الأولى، موصوفة بمشتقات ضمنية. حيث يتم التركيز على الأنظمة الخاضعة لشروط حدودية غير محلية متعددة النقاط. سيتم تحويل النظام التفاضلي إلى صيغة تكاملية مكافئة، والبحث عن نقطة ثابتة في فضاء ليبيغ. سيتم إثبات وجود حل واحد على الأقل باستخدام نظرية شاودر للنقطة الثابتة، حيث سيتم إثبات تراص المؤثر باستخدام معيار كولموغوروف للتراص. علاوة على ذلك، سيتم اشتقاق شروط كافية لوحدانية

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الحل باستخدام مبدأ بناخ للانكماش. أظهرت النتائج النظرية أنه بينما تنتمي المشتقات إلى فضاء ليبيغ، فإن الحلول نفسها تقع في فضاء الدوال المتصلة المطلق. بالإضافة إلى ذلك، نعالج بنجاح -ضمن الإطار ذاته- المسألة المعدلة التي تتضمن شروطاً تكاملية غير محلية لنظام المعادلات المقترنة. وأخيراً، نجري محاكاة عددية باستخدام برنامج ماتلاب لدعم نتائجنا النظرية.

الكلمات المفتاحية: نظام غير خطي مقترن، شرط غير محلي، نظرية شاودر للنقطة الثابتة، وجود ووحدانية الحل، معيار كولموغوروف للتراص، ماتلاب

1. Introduction

Coupled systems of differential equations are fundamental to modeling dynamic processes in various scientific areas such as chemical kinetics, neural networks, and ecological systems. Implicit differential equations, in which the highest-order derivative is not explicitly isolated, often occur in the modeling of constrained physical systems. However, the analysis of these implicit systems is difficult because standard explicit numerical and analytical methods cannot be directly used to follow the evolution of their state variables.

Recently, researchers have started exploring the area of differential equations having non-local boundary conditions as compared to traditional differential equations with initial value problems [1–4, 6–10, 12, 15–21]. In contrast to the classical concept of conditions, which depends upon only one point, the concept of non-local conditions considers more points in their calculations. The non-local concept is considered a more real model of physical problems, as local models cannot deal with "memory or heritance" problems. Many of those articles have studied the solution behavior of an implicit system having multi-point or fractional nonlocal constraints.

Motivated by these developments, the present study contributes to the current literature by investigating a more generalized parameterized coupled implicit system. In particular, we fix $T \in (0,\infty)$ and consider the nonlinear coupled system of two 1st-order implicit ODE

$$\dot{u}(\tau) = \varphi\big(\tau, \dot{v}(\tau)\big), \quad a.e. \quad \tau \in (0,T], \tag{1}$$

$$\dot{v}(\tau) = \psi\big(\tau, \dot{u}(\tau)\big), \quad a.e. \quad \tau \in (0,T] \tag{2}$$

subject to the multipoint nonlocal conditions

$$u(0) + \sum_{\iota=1}^{n} \sigma_\iota\, u(\eta_\iota) = u_o\,, \qquad \eta_\iota \in (0,T], \tag{3}$$

$$v(0) + \sum_{\kappa=1}^{m} \omega_\kappa\, v(\theta_\kappa) = v_o\,, \qquad \theta_\kappa \in (0,T], \tag{4}$$

where φ and ψ are given real-valued functions, σ_ι and ω_κ are positive real parameters for $\iota \in \{1,2,\dots,n\}$ and $k \in \{1,2,\dots,m\}$, and $u_o, v_o \in \mathbb{R}$.

The most important aim of this study is to analyze the existence and uniqueness of solutions, as well as their continuous dependency on the parameters of problem (1)–(4) under suitable hypotheses on the nonlinear functions φ and ψ. This can be achieved by reformulating the system into an equivalent functional setting via a fixed-point framework. In this context, our problem shall be first presented in terms of fixed points for the derivatives; namely, $\xi = \dot{u}$ and $\zeta = \dot{v}$ lying in a Lebesgue space. Of course, the integrability of those derivatives will ensure that the state variables u and v are continued in the space of absolutely continuous functions. We would like to make it clear that this change of variables is crucial because, regardless of the non-smoothness of the derivatives, the solution will still be continuous, hence imparting physical meaning to the nonlocal conditions.

In order to derive some sufficient conditions for the existence of solutions under weak measurability and integrability restrictions, we have employed Schauder's principle along with the Kolmogorov compactness criterion. Additionally, certain contractivity conditions are developed to guarantee the uniqueness of the solution. As an application of our theoretical results, a more general case of our problem will be further

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investigated by relaxing the discrete conditions (3) and (4) into their continuity of time averages. More precisely, the coupled differential equations (1) and (2) associated to the nonlocal integral conditions

$$u(0) + \int_0^\tau u(t)\,dt = u_o\,, \tag{5}$$

$$v(0) + \int_0^\tau v(t)\,dt = v_o \tag{6}$$

would have the same conclusions under identical structural conditions. Finally, a numerical example is provided to illustrate the analytical findings, accompanied by MATLAB simulations to visually confirm our results**.**

2. Preliminaries

Let us begin by recalling some basic notations and well-known facts are necessary throughout the forthcoming analysis of this manuscript. Fix $0 < T < \infty$ and denote by $\mathcal{I}$ the closed interval $[0,T]$ of the real space $\mathbb{R}$. As customary, $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}$ and the Euclidean norm on $\mathbb{R}$ is denoted by $|\cdot|$. The space $L^1(\mathcal{I})$ stands for the set of all functions $\xi\colon \mathcal{I} \to \mathbb{R}$ which are Lebesgue measurable and such that $\int_{\mathcal{I}} |\xi| d\mathcal{L}$ is finite. We denote the associated norm by$\|\xi\|_{L^1}$. Of course, the L^1-space forms a Banach space for the standard norm $\|\xi\|_{L^1} = \int_{\mathcal{I}} |\xi| d\mathcal{L}$. One of the spaces we consider is $\mathbb{L}^1 := L^1(\mathcal{I};\mathbb{R}^2)$ which consists of all couples $(\xi,\zeta)\colon \mathcal{I} \to \mathbb{R}^2$ of the functions $\xi,\zeta \in L^1(\mathcal{I})$ that are Lebesgue integrable. This $\mathbb{L}^1$-space forms a Banach space when endowed with the norm $\|(\xi,\zeta)\|_{\mathbb{L}^1} = \|\xi\|_{L^1} + \|\zeta\|_{L^1}$. Another space which naturally appears is $AC(\mathcal{I})$, the set of all paths $u\colon \mathcal{I} \to \mathbb{R}$ that are absolutely continuous. We recall that a function $u \in AC(\mathcal{I})$ if and only if it can be represented in the form

$$u(\tau) = u(0) + \int_0^\tau \xi(t)\,dt \quad \text{for all } t \in \mathcal{I}\,,$$

where $\xi = \dot{u} \in L^1(\mathcal{I})$ is its weak derivative, defined almost everywhere (a.e.) with respect to the Lebesgue measure $\mathcal{L}$. Under this framework, $AC(\mathcal{I})$ forms a Banach space when endowed with the norm $\|u\|_{AC} = |u(0)| + \|\dot{u}\|_1$. We denote by $\mathbb{AC} := AC(\mathcal{I};\mathbb{R}^2)$, the set of all couples $(u,v)\colon \mathcal{I} \to \mathbb{R}^2$ of the absolutely continuous paths $u, v \in AC(\mathcal{I})$, equipped with the norm

$$\|(u,v)\|_{\mathbb{AC}} = \|u\|_{AC} + \|v\|_{AC}\,.$$

We also recall the following concepts that play a key role in establishing our main results.

Theorem 2.1 (The $\mathbb{L}^1$–Kolmogorov criterion [5, 14]**).** *A bounded subset $\mathbb{X} \subseteq \mathbb{L}^1$ is relatively compact if and only if the uniform convergence $(\xi_\varepsilon,\zeta_\varepsilon) \to (\xi,\zeta)$ holds in the $\mathbb{L}^1$-sense over $\mathbb{X}$ as $\varepsilon \to 0$, where*

$$\xi_\varepsilon(\tau) = \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} \xi(t)\,dt\,,\quad \zeta_\varepsilon(\tau) = \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} \zeta(t)\,dt \quad for\ every \quad 0 < \tau < T - \varepsilon.$$

Theorem 2.2 (Schauder's principle [11]**).** *Let $\mathbb{C}$ be a nonempty, closed, and convex subset of a Banach space $\mathbb{B}$. If the mapping $\mathcal{A}\colon \mathbb{C} \to \mathbb{C}$ is continuous and compact in $\mathbb{B}$, then there is at least one fixed point in $\mathbb{C}$.*

3. Existences of the solutions

Before stating our main results, let us first recast the nonlocal coupled problem (1)–(4) by setting

$$\xi(\tau) = \dot{u}(\tau)\,,\qquad \zeta(\tau) = \dot{v}(\tau) \quad \text{for} \quad \tau \in \mathcal{I}\backslash\{0\}$$

It turns out that the coupled differential equations (1) and (2) can be rewritten down as

$$\xi(\tau) = \varphi\big(\tau,\zeta(\tau)\big),\quad \tau \in \mathcal{I}\backslash\{0\} \tag{7}$$

$$\zeta(\tau) = \psi\big(\tau,\xi(\tau)\big),\quad \tau \in \mathcal{I}\backslash\{0\} \tag{8}$$

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Next, we consider the coupled functional equations (7) and (8) under the following constraints:

(C1) The functions $\varphi, \psi : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable in $\mathcal{I}$ for everywhere in $\mathbb{R}$, and continuous in $\mathbb{R}$ for Lebesgue almost everywhere in $\mathcal{I}$.

(C2) There exist functions $\mu, \nu \in L^1(\mathcal{I})$ such that $|\varphi(\tau,\cdot)| \leq \mu(\tau)$ and $|\psi(\tau,\cdot)| \leq \nu(\tau)$ for all $\tau \in \mathcal{I}$.

(C3) There exist constants $M, N > 0$ such that $\int_{\mathcal{I}} \mu(\tau)d\tau \leq M$ and $\int_{\mathcal{I}} \nu(\tau)d\tau \leq N$.

We are now ready to state:

Lemma 3.1. *Suppose all (C1)–(C3) hold. Then, there exists at least one integrable solution in* $\mathbb{L}^1$ *satisfies the coupled functional system (7)–(8).*

Proof. Let $\chi = (\xi, \zeta)^t \in \mathbb{X}$, where

$$\mathbb{X} = \{(\xi, \zeta) \in \mathbb{L}^1 : \|\xi\|_{L^1} \leq M,\ \|\zeta\|_{L^1} \leq N \text{ for some } M, N \geq 0\}$$

is a nonempty, convex and closed subset of $\mathbb{L}^1$. Clearly, $\|\chi\|_{\mathbb{L}^1} < \infty$. We define the operator $\mathcal{A}$ over $\mathbb{X}$ by

$$\mathcal{A}\chi(\tau) = \big(\Phi\zeta(\tau), \Psi\xi(\tau)\big)^t, \qquad \tau \in \mathcal{I}\backslash\{0\}, \tag{9}$$

where the coupling component operators are given by

$$\Phi\zeta(\tau) = \varphi\big(\tau, \zeta(\tau)\big), \qquad \Psi\xi(\tau) = \psi\big(\tau, \xi(\tau)\big), \quad \tau \in \mathcal{I}\backslash\{0\}.$$

Utilizing the constraints (*C1*) to (*C3*), one obtains the following uniform bounds

$$\|\Phi\zeta\|_{L^1} = \int_0^t \big|\varphi\big(s, \zeta(s)\big)\big| ds \leq \int_0^t \mu(s)ds \leq M,$$
$$\|\Psi\xi\|_{L^1} = \int_0^t \big|\psi\big(s, \xi(s)\big)\big| ds \leq \int_0^t \nu(s)ds \leq N$$

which lead to $(\Phi\zeta, \Psi\xi)^t \in \mathbb{X}$. This showing that $\mathcal{A}$ maps $\mathbb{X}$ into itself (that is; $\mathcal{A}(\mathbb{X}) \subseteq \mathbb{X}$). Furthermore, since $\varphi(\cdot, \zeta), \psi(\cdot, \xi) \in L^1(\mathcal{I})$ via (C2) and $\varphi(\tau,\cdot), \psi(\tau,\cdot) \in C(\mathbb{R})$ via (*C1*), we infer that the continuity of coupling map (Φ, Ψ) over $\mathbb{X}$ yields that the operator $\mathcal{A}: \mathbb{X} \to \mathbb{X}$ is continuous in $\mathbb{L}^1$. All that remains to be shown is the relative compactness of $\mathcal{A}(\mathbb{X})$ in $\mathbb{L}^1$ according to Theorem 2.1. To this end, we observe that the boundedness of $\mathbb{X} \subseteq \mathbb{L}^1$ implies that $(\Phi, \Psi)(\mathbb{X})$ is bounded in $\mathbb{L}^1$, so that $\mathcal{A}(\mathbb{X}) \subseteq \mathbb{L}^1$ is uniformly bounded. We next claim that

$$(\mathcal{A}\chi)_\varepsilon \to \mathcal{A}\chi \text{ uniformly in } \mathbb{L}^1 \text{ for every } \chi \in \mathbb{X} \text{ as } \varepsilon \to 0.$$

To verify this, we let $\chi_\varepsilon = (\xi_\varepsilon, \zeta_\varepsilon)^t \in \mathbb{X}$ such that $\chi_\varepsilon \to \chi$ as $\varepsilon \to 0$ and estimate the identity

$$\|(\mathcal{A}\chi)_\varepsilon - \mathcal{A}\chi\|_{\mathbb{L}^1} = \|(\Phi\zeta)_\varepsilon - \Phi\zeta\|_{L^1} + \|(\Psi\xi)_\varepsilon - \Psi\xi\|_{L^1}. \tag{10}$$

Evaluating the first L^1–norm on the right–hand side, one has

$$\|(\Phi\zeta)_\varepsilon - \Phi\zeta\|_{L^1} = \int_0^\tau \left|\frac{1}{\varepsilon}\int_s^{s+\varepsilon} \Phi\zeta(t)\,dt - \Phi\zeta(s)\right| ds$$
$$\leq \int_0^\tau \frac{1}{\varepsilon}\int_s^{s+\varepsilon} \big|\varphi\big(t, \zeta(t)\big) - \varphi\big(s, \zeta(s)\big)\big|\,dt\,ds.$$

By setting $t = s + r$ with $r \in [0, \varepsilon]$ and applying Fubini's theorem one gets

$$\|(\Phi\zeta)_\varepsilon - \Phi\zeta\|_{L^1} \leq \int_0^\tau \frac{1}{\varepsilon}\int_0^\varepsilon \big|\varphi\big(s + r, \zeta(s + r)\big) - \varphi\big(s, \zeta(s)\big)\big|\,dr\,ds$$
$$\leq \frac{1}{\varepsilon}\int_0^\varepsilon \int_0^\tau \big|\varphi\big(s + h, \zeta(s + r)\big) - \varphi\big(s, \zeta(s)\big)\big|\,ds\,dr.$$

As for the second L^1–norm, it can be computed analogously as follows

$$\|(\Psi\xi)_\varepsilon - \Psi\xi\|_{L^1} = \int_0^\tau \left|\frac{1}{\varepsilon}\int_s^{s+\varepsilon} \Psi\xi(t)\,dt - \Psi\xi(s)\right| ds$$
$$\leq \int_0^\tau \frac{1}{\varepsilon}\int_s^{s+\varepsilon} |\psi(t, \xi(t)) - \psi(s, \xi(s))|\,dt\,ds.$$

Since $\varphi(\cdot, \zeta), \psi(\cdot, \xi) \in L^1(\mathcal{I})$ by (*C2*), the Lebesgue differentiation theorem ensures that for every Lebesgue point $s \in (0, T]$, the following integral means vanish pointwise almost everywhere

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$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{s}^{s+\varepsilon}\left|\varphi\big(t,\zeta(t)\big)-\ \varphi\big(s,\zeta(s)\big)\right|dt=0\,,$$
$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{s}^{s+\varepsilon}\left|\psi\big(t,\xi(t)\big)-\psi\big(s,\xi(s)\big)\right|dt=0\,.$$

Indeed, since $0\le r\le\varepsilon$, taking the limit as $\varepsilon\ \to\ 0$ forces r tends to zero, which directly gives

$$E[\varphi\circ\zeta](r):=\int_{0}^{\tau}\left|\varphi\big(s+r,\zeta(s+r)\big)-\ \varphi\big(s,\zeta(s)\big)\right|ds\xrightarrow[r\to 0]{}0,$$
$$E[\psi\circ\xi](r):=\int_{0}^{\tau}\left|\psi\big(s+r,\xi(s+r)\big)-\psi\big(s,\xi(s)\big)\right|ds\xrightarrow[r\to 0]{}0.$$

If one defines $E_1(\varepsilon):=\max_{0\le r\le\varepsilon}\{E[\varphi\circ\zeta](r)\}$ and $E_2(\varepsilon):=\max_{0\le r\le\varepsilon}\{E[\psi\circ\xi](r)\}$ as their uniform bounds, it follows from the Lebesgue dominated convergence theorem [13, 14] that

$$\frac{1}{\varepsilon}\int_{0}^{\varepsilon}E[\varphi\circ\zeta](r)\,dr\le E_1(\varepsilon)\xrightarrow[\varepsilon\to 0]{}0\,,$$
$$\frac{1}{\varepsilon}\int_{0}^{\varepsilon}E[\psi\circ\xi](r)\,dr\le E_2(\varepsilon)\xrightarrow[\varepsilon\to 0]{}0$$

which confirms the required uniform equicontinuity of $(\Phi,\Psi)(\mathbb{X})$ in the mean (due to Theorem 2.1). Consequently, the $\mathbb{L}^1$–norm of the difference $(\mathcal{A}\chi)_\varepsilon-\mathcal{A}\chi$ in (10) converges uniformly to zero over the bounded set $\mathbb{X}$ as ε goes to zero, which justifies our claim above in the $\mathbb{L}^1$–sense. The proof is completed by the noticing that, in view of Theorem 2.2, as the continuous and compact mapping $\mathcal{A}:\mathbb{X}\to\mathbb{X}$ possesses a fixed point $\chi\in\mathbb{X}$, we may thus deduce that the coupling map (Φ,Ψ) has the desired integrable solution in $\mathbb{L}^1$ which is $(\Phi\zeta,\Psi\xi)^t=(\xi,\zeta)^t$ satisfying the equations (7) and (8).

□

The next theorem basically states that an explicit formula for a solution to the nonlocal coupled problem (1)–(4), which corresponds to the $\mathbb{L}^1$–solution of the coupled functional system (7)–(8), is; indeed, absolutely continuous over $\mathcal{I}$.

Theorem 3.2. *Let* $\sum_{\iota=1}^{n}\sigma_\iota\neq-1$ *and* $\sum_{\kappa=1}^{m}\omega_\kappa\neq-1$. *If the assertation of Lemma 3 holds, then the nonlocal coupled problem (1)–(4) admits at least one absolutely continuous solution in* $\mathbb{AC}$.

Proof. Our knowledge that $\xi(\tau)=\dot{u}(\tau),\zeta(\tau)=\dot{v}(\tau)$ allows us to integrate the both identities from 0 into τ to get

$$u(\tau)=\ u(0)+\int_{0}^{\tau}\xi(t)\,dt\,,\qquad v(\tau)=\ v(0)+\int_{0}^{\tau}\zeta(t)\,dt\,.$$

Inserting the nonlocal conditions (3) and (4) respectively into these integral equations leads to

$$u(\tau)=\ u_o\ -\sum_{\iota=1}^{n}\sigma_\iota\,u(\eta_\iota)+\int_{0}^{\tau}\xi(t)\,dt,\quad v(\tau)=\ v_o-\sum_{\kappa=1}^{m}\omega_\kappa\,v(\theta_\kappa)+\int_{0}^{\tau}\zeta(t)\,dt.\tag{10}$$

Plugging $\tau=\eta_\iota$ and $\tau=\ \theta_\kappa$ into the respect equations in (10), one sees that

$$u(\eta_\iota)=\ u_o\ -\sum_{\iota=1}^{n}\sigma_\iota\,u(\eta_\iota)+\int_{0}^{\eta_\iota}\xi(t)\,dt,\quad v(\theta_\kappa)=\ v_o-\sum_{\kappa=1}^{m}\omega_\kappa\,v(\theta_\kappa)+\int_{0}^{\theta_\kappa}\zeta(t)\,dt.\tag{11}$$

Comparing the equations in (10) and (11) in both respects, this gives

$$u(\eta_\iota)=\ u(\tau)+\int_{0}^{\eta_\iota}\xi(t)\,dt-\int_{0}^{\tau}\xi(t)\,dt,\quad v(\theta_\kappa)=\ v(\tau)+\int_{0}^{\theta_\kappa}\zeta(t)\,dt-\int_{0}^{\tau}\zeta(t)\,dt.\tag{12}$$

Then substituting the equations in (12) into those in (10); respectively, one obtains the expressions

$$u(\tau)=\ u_o\ -\sum_{\iota=1}^{n}\sigma_\iota\left[u(\tau)+\int_{0}^{\eta_\iota}\xi(t)\,dt\ -\int_{0}^{\tau}\xi(t)\,dt\right]+\int_{0}^{\tau}\xi(t)\,dt\ ,$$
$$v(\tau)=\ v_o-\sum_{\kappa=1}^{m}\omega_\kappa\left[v(\tau)+\int_{0}^{\theta_\kappa}\zeta(t)\,dt\ -\int_{0}^{\tau}\zeta(t)\,dt\right]+\int_{0}^{\tau}\zeta(t)\,dt$$

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which implies

$$u(\tau) = (1+\sigma)^{-1}\left[u_o - \sum_{\iota=1}^{n}\sigma_\iota \int_0^{\eta_\iota}\xi(t)\,dt\right] + \int_0^{\tau}\xi(t)\,dt\,, \tag{13}$$

$$v(\tau) = (1+\omega)^{-1}\left[v_o - \sum_{\kappa=1}^{m}\omega_\kappa \int_0^{\theta_\kappa}\zeta(t)\,dt\right] + \int_0^{\tau}\zeta(t)\,dt, \tag{14}$$

where $\sigma = \sum_{\iota=1}^{n}\sigma_\iota$ and $\omega = \sum_{\kappa=1}^{m}\omega_\kappa$. From Lemma 3.1 along with the integral forms (13) and (14), we conclude the existences of at least one absolutely continuous solution to problem (1)–(4); namely, $w = (u,v)^t \in \mathbb{AC}$ such that $u, v \in AC(\mathcal{I})$ with $\|w\|_{\mathbb{AC}} = \|u\|_{AC} + \|v\|_{AC}$ and this ends the proof.

□

As a direct consequence of previous results, the standard equivalence between the nonlocal coupled problem (1)–(4) and the coupled system of integral representations (13) and (14) reads as

Corollary 3.3. *A couple of functions $w = (u,v)^t$ is a solution of the nonlocal coupled problem (1)–(4) in $\mathbb{AC}$ if and only if it can be represented by the coupled integral equations (13) and (14); respectively.*

Proof. Obviously, the direct implication follows immediately from Theorem 4. We next show the reverse by differentiating both sides of (13) and (14) in τ to get

$$\dot{u}(\tau) = \xi(\tau) = \varphi\big(\tau,\zeta(\tau)\big), \qquad \dot{v}(\tau) = \zeta(\tau) = \psi\big(\tau,\xi(\tau)\big)\,.$$

Then we plug $\tau = 0$ into both equations (13) and (14), which have the forms

$$u(0) = (1+\sigma)^{-1}\left[u_o - \sum_{\iota=1}^{n}\sigma_\iota \int_0^{\eta_\iota}\xi(t)\,dt\right], \tag{15}$$

$$v(0) = (1+\omega)^{-1}\left[v_o - \sum_{\kappa=1}^{m}\omega_\kappa \int_0^{\theta_\kappa}\zeta(t)\,dt\right]. \tag{16}$$

Setting $\tau = \eta_\iota$ and $\tau = \theta_\kappa$ in (13) and (14), and multiplying by $\sum_{\iota=1}^{n}\sigma_\iota$ and $\sum_{\kappa=1}^{m}\omega_\kappa$; respectively, gives

$$\sum_{\iota=1}^{n}\sigma_\iota\, u(\eta_\iota) = (1+\sigma)^{-1}\sum_{\iota=1}^{n}\sigma_\iota\left[u_o - \sum_{\iota=1}^{n}\sigma_\iota \int_0^{\eta_\iota}\xi(t)\,dt\right] + \sum_{\iota=1}^{n}\sigma_\iota \int_0^{\eta_\iota}\xi(t)\,dt, \tag{17}$$

$$\sum_{\kappa=1}^{m}\omega_\kappa v(\theta_\kappa) = (1+\omega)^{-1}\sum_{\kappa=1}^{m}\omega_\kappa\left[v_o - \sum_{\kappa=1}^{m}\omega_\kappa \int_0^{\theta_\kappa}\zeta(t)\,dt\right] + \sum_{\kappa=1}^{m}\omega_\kappa \int_0^{\theta_\kappa}\zeta(t)dt. \tag{18}$$

Combining the initial solutions (15) and (16) with corresponding nonlocal equations (17) and (18), we retrieve the nonlocal conditions (3) and (4). Thus, the nonlocal coupled problem (1)–(4) has at least one solution $w = (u,v)^t \in \mathbb{AC}$, which gives the desired conclusion.

□

4. Uniqueness of the solution

For the uniqueness result, we consider the nonlocal problem (1)–(4) under the further constraints:

(C4) For each $\xi, \zeta \in \mathbb{R}$, the mappings $\tau \mapsto \varphi(\tau,\xi)$ and $\tau \mapsto \psi(\tau,\zeta)$ are Lebesgue measurable on $\mathcal{I}$,
(C5) There exist constants $\ell_1, \ell_2 > 0$ such that for all $\tau \in \mathcal{I}$

$$\big|\varphi(\tau,\xi) - \varphi(\tau,\breve{\xi})\big| \le \ell_1\big|\xi - \breve{\xi}\big|\,, \quad \big|\psi(\tau,\zeta) - \psi(\tau,\breve{\zeta})\big| \le \ell_2\big|\zeta - \breve{\zeta}\big| \text{ for every } \xi,\breve{\xi},\zeta,\breve{\zeta} \in \mathbb{R},$$

(C6) $\ell := \max\{\ell_1,\ell_2\} < 1$.

Theorem 4.1. *Suppose all (C4)–(C6) hold. Then, there exists at most one solution in $\mathbb{AC}$ satisfies the nonlocal coupled problem (1)–(4).*

Proof. Let $\chi = (\xi,\zeta)^t$ and $\breve{\chi} = (\breve{\xi},\breve{\zeta})^t$ be two integrable solutions of the coupled functional system (7)–(8) in $\mathbb{L}^1$. By using the operator definition (9) and employing the corresponding L^1-norms for $\Phi\xi$ and $\Psi\zeta$ over $\mathcal{I}$, we evaluate the subtraction of $\mathcal{A}\chi$ and $\mathcal{A}\breve{\chi}$ in the $\mathbb{L}^1$-norm as follows

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$$\|\mathcal{A}\chi - \mathcal{A}\breve{\chi}\|_{\mathbb{L}^1} = \left\|\Phi\xi - \Phi\breve{\xi}\right\|_{L^1} + \left\|\Psi\zeta - \Psi\breve{\zeta}\right\|_{L^1}$$
$$= \int_0^1 \left|\varphi(\tau,\xi) - \varphi\left(\tau,\breve{\xi}\right)\right| d\tau + \int_0^1 |\psi(\tau,y_1) - \psi(\tau,\breve{y}_1)|\, d\tau\,.$$

Then, by utilizing the Lipschitz inequalities in (*C5*), one obtains

$$\|\mathcal{A}\chi - \mathcal{A}\breve{\chi}\|_{\mathbb{L}^1} \le \ell_1 \int_0^1 \left|\xi(\tau) - \breve{\xi}(\tau)\right| d\tau + \ell_2 \int_0^1 \left|\zeta(\tau) - \breve{\zeta}(\tau)\right| d\tau$$
$$\le \ell_1 \left\|\xi - \breve{\xi}\right\|_{L^1} + \ell_2 \left\|\zeta - \breve{\zeta}\right\|_{L^1}.$$

By setting $\ell := \max\{\ell_1, \ell_2\}$, it follows that the above inequality reduces to

$$\|\mathcal{A}\chi - \mathcal{A}\breve{\chi}\|_{\mathbb{L}^1} \le \ell \left(\left\|\xi - \breve{\xi}\right\|_{L^1} + \left\|\zeta - \breve{\zeta}\right\|_{L^1}\right)$$

which implies the following estimate for the $\mathbb{L}^1$-norm

$$\|\mathcal{A}\chi - \mathcal{A}\breve{\chi}\|_{\mathbb{L}^1} \le \ell \|\chi - \breve{\chi}\|_{\mathbb{L}^1}.$$

Since $\ell < 1$ due to (*C6*), the operator $\mathcal{A}$ is a strict contraction on $\mathbb{L}^1$. Consequently, due to the Banach contraction principle [11, 14], $\mathcal{A}$ admits at most one fixed point, which yields the existence of a single $\chi \in \mathbb{L}^1$. Since $\chi = (\xi, \zeta)^t$ with $\xi, \zeta \in L^1(\mathcal{I})$ is uniquely determined, we infer that the unique initial data $w(0) = (u(0), v(0))^t$ can be explicitly found by (15) and (16). Thus, the uniqueness of ξ, ζ along with $u(0), v(0)$ gives directly the uniqueness of the solution $w = (u, v)^t \in \mathbb{AC}$, as desired. □

5. Nonlocal integral conditions

Let $w = (u, v)^t \in \mathbb{AC}$ be the solution of the nonlocal coupled problem (1)–(4), where $u, v \in AC(\mathcal{I})$. We introduce the two finite partitions of the interval $\mathcal{I}$ satisfying

$$0 < \varrho_0 < \varrho_1 < \cdots < \varrho_n < 1 \quad \text{and} \quad 0 < \delta_0 < \delta_1 < \cdots < \delta_n < 1.$$

By setting the parameter increments $\sigma_\iota := \sigma(\tau_\iota) = \varrho_\iota - \varrho_{\iota-1}$ for $\eta_\iota \in (\varrho_{\iota-1}, \varrho_\iota)$ with $\iota = 1,2,\dots,n$ and $\omega_\kappa := \omega(\tau_\kappa) = \delta_\kappa - \delta_{\kappa-1}$ for $\theta_\kappa \in (\delta_{\kappa-1}, \delta_\kappa)$ with $\kappa = 1,2,\dots,m$, the nonlocal boundary conditions (3) and (4) can be equivalently expressed as the discrete forms

$$u(0) + \sum_{\iota=1}^{n} (\varrho_\iota - \varrho_{\iota-1})\, u(\eta_\iota) = u_o\,, \qquad \eta_\iota \in (\varrho_{\iota-1}, \varrho_\iota)\,,$$
$$v(0) + \sum_{\kappa=1}^{m} (\delta_\kappa - \delta_{\kappa-1})\, v(\theta_\kappa) = v_o\,, \quad \theta_\kappa \in (\delta_{\kappa-1}, \delta_\kappa)\,.$$

Taking the limits respectively in both equations as $n \to \infty$ and $m \to \infty$, one has

$$u(0) + \lim_{n\to\infty} \sum_{\iota=1}^{n} (\varrho_\iota - \varrho_{\iota-1})\, u(\eta_\iota) = u_o\,,$$
$$v(0) + \lim_{m\to\infty} \sum_{\kappa=1}^{m} (\delta_\kappa - \delta_{\kappa-1})\, v(\theta_\kappa) = v_o\,.$$

Consequently, the continuity of $w = (u, v)^t \in \mathbb{AC}$ leads to the convergence of the Riemann sums in last equations and; indeed, they can be reduced directly to the nonlocal integral forms (5) and (6).

Corollary 5.1. *Assume that all constraints of Theorem 3.2 (or, respectively, Theorem 4.1) hold. Then, the coupled differential equations (1) and (2) associated to the nonlocal integral conditions (5) and (6) admit at least (respect. at most) one solution* $w \in \mathbb{AC}$.

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6. Numerical Example

In this section, we provide a numerical example to illustrate the applicability of our theoretical results. Consider the following coupled differential equations

$$\dot{u}(\tau) = \frac{1}{10}\, e^{-\tau} + \frac{1}{15}\sin(\dot{v}(\tau)), \qquad a.e.\ \tau \in (0,1],$$
$$\dot{v}(\tau) = \frac{1}{8}\,\tau^2 + \frac{1}{12}\ln(1 + |\dot{u}(\tau)|), \qquad a.e.\ \tau \in (0,1]$$

subject to the nonlocal boundary conditions

$$u(0) + 0.3u(0.5) + 0.2\,u(1.01) = 4,$$
$$v(0) + 0.4v(0.4) + 0.1\,v(0.9) = 6.$$

Let us first define the nonlinear functions

$$\varphi(\tau, \dot{v}) = \frac{1}{10}\, e^{-\tau} + \frac{1}{15}\sin(\zeta), \quad \psi(\tau, \dot{u}) = \frac{1}{8}\,\tau^2 + \frac{1}{12}\ln(1 + |\xi|).$$

By mapping the system onto the space $\mathbb{L}^1([0,1];\mathbb{R}^2)$ using the state transformations $\xi = \dot{u}$ and $\zeta = \dot{v}$, we obtain the corresponding functional equations

$$\xi(\tau) = \varphi(\tau, \zeta) = \frac{1}{10}\, e^{-\tau} + \frac{1}{15}\sin(\zeta),$$
$$\zeta(\tau) = \psi(\tau, \xi) = \frac{1}{8}\,\tau^2 + \frac{1}{12}\ln(1 + |\xi|).$$

Here, the discrete nonlocal boundary parameters are given by

$$\sigma_1 = 0.3\,, \sigma_2 = 0.2, \eta_1 = 0.5, \eta_2 = 1.01, u_o = 4,$$
$$\omega_1 = 0.4, \omega_2 = 0.1, \theta_1 = 0.4, \theta_2 = 0.9, v_o = 6.$$

Notice first that both functions φ and ψ are clearly Lebesgue measurable with respect to time variable τ (due to the continuity of $e^{-\tau}$ and τ^2 on $[0,1]$ for any fixed $\xi, \zeta \in \mathbb{R}$) and continuous with respect to the state variables ξ and ζ (due to the respect continuity of $\sin(\zeta)$ and $\ln(1 + |\xi|)$ everywhere on $\mathbb{R}$ for any fixed $\tau \in [0,1]$). Moreover, since $|\sin(\zeta)| \le 1$ and $\ln(1 + |\xi|) \le |\xi|$ (due to $|\xi| \ge 0$) for all $\xi, \zeta \in \mathbb{R}$, one can use the triangle inequality to have the following bounds

$$|\varphi(\tau, \zeta)| \le \frac{1}{10}\, e^{-\tau} + \frac{1}{15} = \mu(\tau), \qquad |\psi(\tau, \xi)| \le \frac{1}{8}\,\tau^2 + \frac{1}{12}|\xi| = \nu(\tau).$$

As both μ and ν are time-continuous on$[0,1]$, they are integrable and it follows that $\mu, \nu \in \mathbb{L}^1([0,1])$. In particular, for any $\tau \in [0,1]$, we observe that $e^{-\tau} \le 1$ and $\tau^2 \le 1$, so one may choose the constants $M \ge 1/6$ and $N \ge (1/8) + (1/12)\|\xi\|_1$ such that the following L^1-norms estimates over $[0,1]$

$$\|\varphi\|_{L^1} = \int_0^1 \mu(t)\, dt \le \int_0^1 \left(\frac{1}{10} + \frac{1}{15}\right) dt \le M$$
$$\|\psi\|_{L^1} = \int_0^1 \nu(t)\, dt \le \int_0^1 \left(\frac{1}{8} + \frac{1}{12}|\xi|\right) dt \le N$$

Confirm that φ and ψ fully satisfy constraints ($C1$)–($C3$). Furthermore, since $\sum_{\iota=1}^{2} \sigma_\iota = \sum_{\kappa=1}^{2} \omega_\kappa = 0.5 \ne -1$, we infer from Theorem 3.2 that the given example has at least one solution $w = (u, v)^t$ in $\mathbb{AC}([0,1];\mathbb{R}^2)$.

Next, we turn to the uniqueness of the solution. For any $\xi, \check{\xi}, \zeta, \check{\zeta}$ in$\mathbb{L}^1([0,1])$, we have

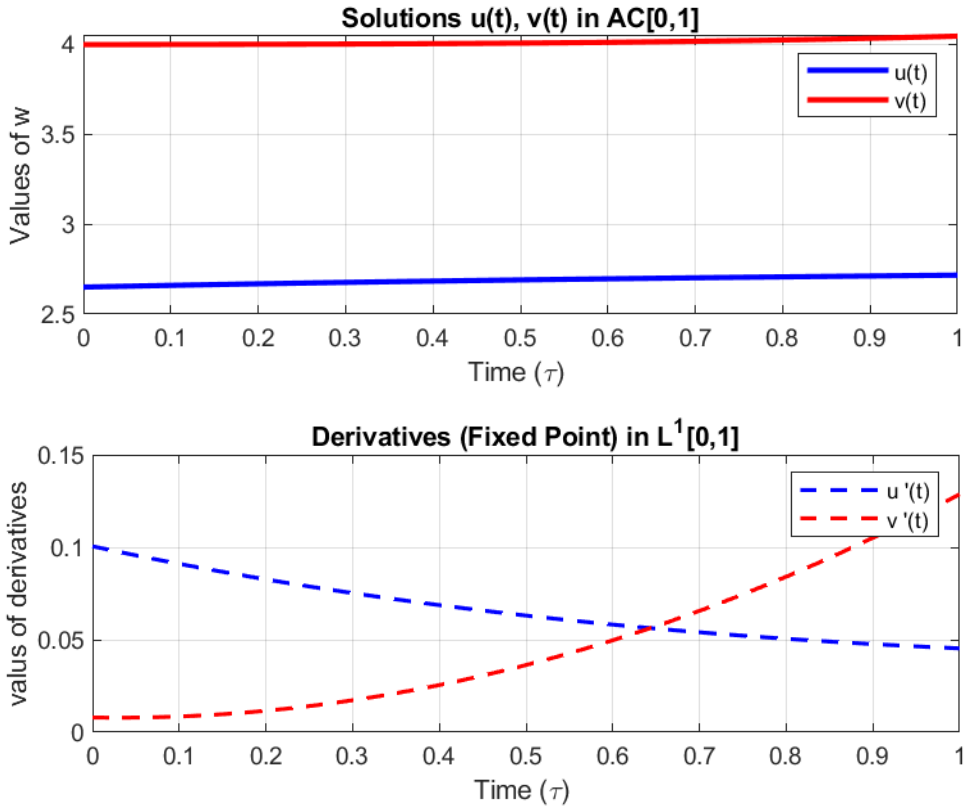
$$\left|\varphi(\tau, \zeta) - \varphi(\tau, \check{\zeta})\right| = \frac{1}{15}\left|\sin(\zeta) - \sin(\check{\zeta})\right| \le \frac{1}{15}\left|\zeta - \check{\zeta}\right|,$$
$$\left|\psi(\tau, \xi) - \psi(\tau, \check{\xi})\right| = \frac{1}{12}\left|\ln(1 + |\xi|) - \ln(1 + |\check{\xi}|)\right| \le \frac{1}{12}\left|\xi - \check{\xi}\right|.$$

Thus, the respective Lipschitz constants are $\ell_1 = 1/15$ and $\ell_2 = 1/12$. It follows that

$$\ell = \max\{\ell_1, \ell_2\} = \frac{1}{12} \approx 0.0833 < 1.$$

Hence, as all constraints ($C4$)–($C6$) are fulfilled, Theorem 6 asserts that the given example has only one solution $w = (u, v)^t \in \mathbb{AC}([0,1];\mathbb{R}^2)$.

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The figures above illustrate the numerical solutions and their corresponding derivative profiles for the nonlinearly coupled differential system given by our example. The upper plot depicts the state trajectories $u(\tau)$ and $v(\tau)$, which display smooth and continuous behavior within the space of absolutely continuous maps $\mathbb{AC}([0,1];\mathbb{R}^2)$. These trajectories strictly fulfill the nonlocal integral-point boundary conditions at the specified evaluation points, achieving the exact target values of 4 and 6. The lower plot illustrates the derivative functions $\dot{u}(\tau)$ and $\dot{v}(\tau)$ residing in the $\mathbb{L}^1[0,1]$ Lebesgue space. The decreasing trend of $\dot{u}(\tau)$ is directly governed by the decaying exponential term $e^{-\tau}$, whereas the quadratic term τ^2 accounts for the monotonic increase of $\dot{v}(\tau)$. The distinct intersection of the two derivative profiles near $\tau \approx 0.65$ reflects the strong nonlinearly coupled interaction of the system, confirming the existence and uniqueness of a stable fixed-point solution as established by the Banach fixed-point theorem.

The visual representations play a crucial role in complementing the analytical proof of existence and uniqueness. The figures above demonstrate that the generated state trajectories belong to the space of absolutely continuous functions $\mathbb{AC}[0,1]$, confirming the existence result. Moreover, the steady and unique profiles of the derivative functions in $\mathbb{L}^1[0,1]$ provide numerical evidence of the contraction mapping principle (Banach fixed-point theorem), guaranteeing that the system possesses a single stable solution satisfying all nonlocal constraints.

7. Conclusion

This paper presents theoretical and numerical study of a nonlinearly coupled first-order differential system with nonlocal boundary conditions. Existence and uniqueness of solutions in $\mathbb{AC}[0,1]$ with derivatives in $\mathbb{L}^1[0,1]$ are proved using the Banach Fixed Point Theorem. Direct numerical simulations in MATLAB validate the theoretical results, demonstrating precise satisfaction of the nonlocal constraints.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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